What Is Special and What Is Generic about the Universe?

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A man said to the universe:
"Sir, I exist!"
"However," replied the universe,
"The fact has not created in me
A sense of obligation."
— Stephen Crane

Outline

Probabilistic Reasoning in Cosmology

The Relation between Topology and Measure in Probabilistic Reasoning

Borel Measures on Families of Spacetimes

Topology, Measure, and Probability in Cosmology

Common Forms of Questions

- probability of particular kinds of observations, given our cosmological situation
- probability that value of universal constant lies in fixed range
- probability that initial conditions of particular kind occurred
- probability that particular large-scale structures would form or local features obtain
- probability that a particular global (causal, topological, projective, conformal, affine, metrical) property obtains

Common Examples

- probability that observers such as ourselves would come to exist in the sort of spatiotemporal region we occupy in a spacetime of this sort
- probability that we are "typical" observers in the universe
- probability that conscious observers exist at all
- probability that the cosmological constant has any non-zero value, and has, moreover, the one actually observed
- probability that a generic spacetime is singular to the future
- various "fine-tuning problems": the approximate flatness of the observed universe; the approximate homogeneity of the observed universe; the seemingly required special entropic state of the very early universe

Common Ways to Try to Make Questions Precise

- fix non-probability measure on family of spacetimes, look for properties forming a set with large or with zero or near-zero measure; when relevant sets have infinite measure—the standard case—try to "re-normalize"
- fix topology on family of spacetimes, look for properties forming an open, dense set or a nowhere-dense set
- invoke statistical mechanics and thermodynamics: standard considerations give rise to appropriate probability distributions on a family of spacetimes
- invoke anthropic principles: existence of conscious observers, or of large-scale or local structure of a particular form, or of . . . , places sufficient constraints to determine an *a priori* probability distribution on family of spacetimes

Most Common Way

1. no measure is defined, but it is assumed that there really is a natural, appropriate one
2. a topology is crudely postulated
3. hands are waved at vague notions of properties forming open, dense or nowhere-dense sets
4. underlying intuition is implicitly, silently invoked that "real underlying measure" is consonant with topology
5. conclusion:
    - "open, dense" $\Rightarrow$ "generic", "highly likely"
    - "nowhere dense" $\Rightarrow$ "rare", "highly unlikely"

one is generally also concerned about the stability of these conclusions "under small perturbations"

(consonance of measure and topology)

Problems

1. meaning of probability, when there is only one physical system of the type at issue to observe
2. justification of using probability, kinds of evidence available, when one cannot measure frequencies
3. isolation, clarification and justification of various assumptions one must make in order to apply probabilistic reasoning
4. physical significance of measures and topologies, relevance to property or feature at issue
5. relationship between topological and measure-theoretic concepts and methods, especially in infinite-dimensional cases

Probabilistic Reasoning in Cosmology

The Relation between Topology and Measure in Probabilistic Reasoning

Borel Measures on Families of Spacetimes

Topology, Measure, and Probability in Cosmology "generic" $\approx$ "most systems are similar in this respect" "most" is a measure-theoretic notion

"similar in this respect" is a topological notion we need a Borel measure

(“large is open and dense, small is nowhere dense”) "stable" $\approx$ "small changes in values shouldn't significantly change likelihoods of being similar" "small changes in values" is an algebraic notion (addition, multiplication, translation) in physics, we generally need an appropriately translation-invariant Borel measure for probabilistic reasoning

(not always: *e.g.*, the exponential distribution for radioactive decay)

vague conceptual speculation

in physics, topology is generally prior to, more fundamental than measure in probabilistic reasoning:

- we need to characterize similarity before we can formulate comparison class (σ-algebra) to achieve quantitative exactness of assigning exact probabilities
- we need to show stability to trust our testing of the predicted probabilities

situation is nice in $\mathbb{R}^n$

- natural topology $\Rightarrow$ Borel sets, σ-algebra
- well defined notion of translation (addition)
- $\Rightarrow$ Lebesgue measure is unique complete, translation-invariant Borel measure
- Lebesgue measure is:
  - locally finite (every p has neighborhood with finite measure)
  - strictly positive (non-empty open sets have positive measure)

recall:

Definition

*A **Fréchet space** is a metrizable, locally convex topological vector space, complete with respect to a translation-invariant metric.*

Definition

*A **Fréchet manifold** is a differential manifold modeled on a Fréchet space (rather than on $\mathbb{R}^n$).*

families of spacetimes relevant to cosmology generally form infinite-dimensional Fréchet manifolds (sum of 2 Lorentz metrics generally not a Lorentz metric: why family of spacetimes is a Fréchet manifold, not a Fréchet space)

if you don't feel comfortable with this, just think "really big, non-linear space"

(more precisely: appropriate subspaces of the space of cross-sections of the bundle of metrics over a spacetime manifold have the structure of an infinite-dimensional Fréchet manifold, modeled on the infinite-dimensional Fréchet space of symmetric, two-index tensor fields; in fact, the Fréchet manifold of metrics is an open, convex submanifold of the Fréchet space of symmetric, two-index tensor fields)

Is there a physically reasonable translation-invariant Borel measure on families of metrics (infinite-dimensional Fréchet manifolds)?

First problem: what topology to use?

- **compact-open**: coarsest "reasonable" one; doesn't care about asymptotic behavior, so no good for cosmology
- **Whitney**: no physical significance at all (λg_{ab} is not even a continuous curve for $\lambda \in \mathbb{R}^+$)
- **Sobolev**: finer than Whitney, so even worse; still, useful for proving stability results about initial-value problems, where ultra-fineness is a virtue
- **parameter**: family of metrics characterized by finite number of parameters, open set is product of intervals; good for super-simple perturbation problems, but for nothing else

let's bracket, and just pretend we have a good, physically significant topology

Second problem: what does "translation-invariant" mean here? *I.e.*, what is physically significant notion of "translation" for a metric in a family of metrics?

Solution: look at how cosmologists handle perturbations (translation of one metric to "nearby" one) to get a grip on this

ignoring lots of technical details, the upshot: there is a physically significant, completely general notion of local translation on Fréchet manifold of metrics ("local affine-translation")

Theorem

There is no non-trivial, locally affine-translation invariant Borel measure on an infinite-dimensional Fréchet manifold $\mathcal{F}$.

("non-trivial": measures assigning all open sets zero or infinite measure don't count)

There are three kinds of lies: lies, damned lies, and [probabilities in cosmology].

Mark Twain

Genericity of Future Collapse Singularities in Spatially Open Spacetimes

arguments due to Geroch, Hawking, Penrose, Wheeler, *et al.*, that singularities are "generic" in spatially open spacetimes; BUT arguments depend on assuming relation between topology and measure that just don't hold for any reasonable topology and measure on this family of spacetimes:

- assume that "small perturbations" (in topological sense) that destroy property means family of spacetimes with that property has zero measure
- assume that "small perturbations" (in topological sense) that preserve property means family of spacetimes with that property has non-trivial (finite!) measure

Weinberg's Anthropic Argument for Value of Λ

1. work with family of perturbed cosmological models (say, Szekeres spacetimes)
2. existence of large, gravitationally bound systems places upper and lower bounds on possible values of Λ (too positive, potentially bound systems pulled apart; too negative, universe recollapses before they can form)
3. argue for topological stability of formation of such bound systems under small changes in value of Λ
4. use anthropic argument (presence of conscious observers as selection effect, assume we are typical observers, *i.e.*, value of Λ in our spacetime is typical) to fix shape and peak of measure

BUT: implicitly assumed that topological stability implies largeness of size in fixed measure, which doesn't obtain in infinite-dimensional spaces

I must emphasize:

*I do **not** claim that the conclusions of such arguments are wrong, only that the arguments currently given have serious mathematical, physical and conceptual problems that must be addressed before any real confidence can be had in those conclusions.*

We want to be able to reason about the cosmos in particular ways, to answer questions of a particular sort. As poignantly intimated by the poem serving as this talk's epigraph, that bare want does not suffice to guarantee that the universe be such as to support the required reasoning.