Interaction and Evolution in Classical Mechanics and in Quantum Mechanics, and Indefinite Causal Orders

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I will draw out and make precise some differences in the mathematical representations of "interaction" and "evolution" in classical mechanics and quantum mechanics, and begin to draw some conceptual lessons from that analysis.

In particular, I will propose that the lack of a clear distinction between interaction and evolution in QM, and the concomitant "decoupling" of quantum dynamics from background spacetime structure, jointly suggest a natural way to conceive of indefinite causal order in sum, classical (Newtonian) mechanics and Lagrangian mechanics:

1. characterize "interaction" and "evolution" as naturally distinguished mathematically, physically and conceptually by the dynamics alone (and so intrinsic to it)
2. ⇒ natural distinction between configuration and momentum
3. ⇒ natural characterization of free evolution
4. ⇒ construction of (*a priori* relation to) spacetime geometry
5. clean separation of system's degrees of freedom from environment's

in sum, Hamiltonian mechanics:

1. NO natural distinction between interaction and evolution intrinsic to dynamics
2. ⇒ NO natural distinction between configuration and momentum
3. ⇒ NO natural characterization of free evolution
4. ⇒ NO construction of (*a priori* relation to) spacetime geometry
5. still, QUALIFIED clean separation of system's degrees of freedom from environment's

in sum, in quantum mechanics:

1. NO natural distinction between interaction and evolution intrinsic to dynamics
2. $\Rightarrow$ NO natural distinction between configuration and momentum
3. $\Rightarrow$ NO natural characterization of free evolution
4. $\Rightarrow$ NO construction of (*a priori* relation to) spacetime geometry
5. NO clean separation of system's degrees of freedom from environment's

then "indefinite causal order":

- different sets of Qs and Ps, each satisfying canonical commutation relation
- each set determined by a different null-cone structure
- the different null-cone structures related by geometrical operations that induce algebraic relations between the sets of Qs and Ps
- relation to process matrices?

Outline

classical system, rough first pass

1. space of states $\mathcal{S}$ is even-dimensional manifold
2. evolution governed by Newton's Second Law (family of dynamical vector fields $\mathfrak{D}$)
3. "interaction": applying an "external force" to the system ("intervention")
4. distinguished "free" evolution ("isolation" = no external force)

Newton's Second Law

free particle

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \mathbf{0}\end{aligned} \tag{1}$$

dynamical vector field has components: $(\mathbf{v},\, \mathbf{0})$

with interaction turned on ("hit it with a stick")

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \mathbf{F}_{\text{stick}}\end{aligned} \tag{2}$$

dynamical vector field: $(\mathbf{v},\, \mathbf{F}_{\text{stick}})$
interaction vector field: $(\mathbf{0},\, \mathbf{F}_{\text{stick}})$

- dynamical evolution represented by vector field that always has $\mathbf{v}$ for the component of the $\dfrac{\partial}{\partial \mathbf{x}}$ part
- interaction represented by vector field that always has $\mathbf{0}$ for the component of the $\dfrac{\partial}{\partial \mathbf{x}}$ part: it "points only in velocital directions" (an acceleration, instantaneous change in velocity, not in position)
- any magnitude of force in any direction can be applied to any system (in principle)
- multiple simultaneously applied forces sum vectorially

Definition

A classical system *is fully characterized by:*

1. *space of states is even-dimensional manifold;*
2. *the family of interaction vector fields* $\mathfrak{I}$ *has the structure of a vector space, whose integral curves determine a family of disjoint submanifolds of* $\mathcal{S}$ *each half its dimension and jointly foliating it (and which satisfies a few more technical conditions);*
3. *the family of dynamical vector fields* $\mathfrak{D}$ *has the structure of an affine space modeled on* $\mathfrak{I}$*.*

(the "fully characterized by" is justified by the theorem soon to come) "obvious" from formulation of Newton's Second Law and the representations of $\mathfrak{D}$ and $\mathfrak{I}$:

1. crudely, if I can hit a particle with two different sticks, I can hit it with both at the same time
2. the difference between any two evolutions is just the force I need to apply to one to get to the other
3. this yields classification of quantities as "configurative" and "velocital"—if two points of $\mathcal{S}$ are connected by an integral curve of an interaction vector field, then those states "have the same configuration", and a quantity is configurative iff it's constant on all corresponding submanifolds of $\mathcal{S}$
4. free dynamical vector field distinguished by fact that only configurative quantities change

(one can make this all precise and rigorous, for finite-dimensional systems AND fields with infinite-dimensional spaces of state) conceptually, the family of possible interactions allows one to distinguish "configuration" from "velocity" quantities in a principled way:

- interactions directly modify the system's evolutions only in "velocital directions", *i.e.*, as generalized "accelerations"
- "velocity" quantities are always dynamical derivatives of "configuration" ones (in sense of affine structure on family of dynamical vector fields)

$\Longrightarrow$ *the physical meaning of "configuration": those quantities encode the possible interactions the system can have with its environment*

Lagrangian mechanics

Theorem (Curiel 2014)

Given classical system with space of states $\mathcal{S}$ and families of vector fields $\mathfrak{D}$ and $\mathfrak{I}$:

1. *one can reconstruct the system's configuration space $\mathcal{C}$ from the algebraic structure of $\mathfrak{I}$;*
2. *$\mathcal{S}$ is canonically isomorphic to $T\mathcal{C}$ (using the distinguished free dynamical vector field to define the isomorphism);*
3. *$\mathfrak{D}$ is then isomorphic to the family of vector fields on $T\mathcal{C}$ representing all solutions to the Euler-Lagrange equation ("second-order vector fields", i.e., lifts of vector fields from configuration space to the tangent bundle);*
4. *$\mathfrak{I}$ is isomorphic to the family of vector fields on $T\mathcal{C}$ representing generalized forces ("vertical vector fields", pointing straight up and down the fibers);*
5. *in particular, the vertical vector fields have the structure of a vector space, and the second-order vector fields the structure of an affine space modeled on it.*

the geometry of the Euler-Lagrange equation

Theorem (Curiel 2014)

If a manifold $\mathcal{M}$ admits a formulation of the Euler-Lagrange equation, in the sense of having an operator mapping scalar fields to a family of vector fields $\mathfrak{D}$ having the appropriate structure ("integrable, complete almost-tangent structure"), then:

1. *it is diffeomorphic to a tangent bundle;*
2. *that operator allows one to construct configuration space $\mathcal{C}$, and to construct a canonical isomorphism between $\mathcal{M}$ and $T\mathcal{C}$ taking $\mathfrak{D}$ to the second-order vector fields;*
3. *in particular $\mathfrak{D}$ is an affine space modeled on the vector space of vector fields on $\mathcal{M}$ mapped to the vertical vector fields on $T\mathcal{C}$.*

classical mechanics just is
Lagrangian mechanics

Lagrangian mechanics defines and enforces:

1. configuration and velocity, and the difference between them
2. the fixed kinematical relation between them (the latter is the "dynamical derivative" of the former)[1]
3. a notion of interaction ("interventions") distinct from dynamical evolution
4. a distinguished notion of "isolation"

1. Technically:

$$\begin{aligned} J^n{}_a \, (\mathsf{d}q)_n &= 0 \\ J^n{}_a \, (\mathsf{d}v)_n &= \left(\frac{\partial}{\partial q}\right)^a \end{aligned} \tag{3}$$

where $J^a{}_b$ is the canonical almost-tangent structure on the tangent bundle, and (q, v) are natural coordinates. This is the geometrical, generalized version of $\dot{\mathbf{x}} = \mathbf{v}$.

if I know only the space of states as an abstract manifold and how to solve the Euler-Lagrange equation (or Newton's Second Law), I can reconstruct **everything else**

$\Longrightarrow$ *the dynamics* <u>*by itself*</u> *intrinsically and automatically*

- *defines and encodes the difference between "configuration" and "velocity"*
- *and, correlatively, between "evolution" and "interaction",*
- *and uniquely determines "isolation" ("free evolution")*

this is what I mean by "classical system":

1. mathematical and conceptual distinction between "configuration" and "velocity" encoded in the dynamics
2. mathematical and conceptual distinction between "evolution" and "interaction" encoded in the difference between configuration and velocity
3. and so a naturally distinguished "free" evolution

always a clean separation between a system's degrees of freedom and those of its environment:

- no matter the interaction, I can determine the system's evolution without knowing any details about the environment's degrees of freedom or its evolution
- I treat the environment like a black box, everything relevant encoded in interaction vector field

(akin to Don Howard's reconstruction of Bohr's notion of "classical", made rigorous; difficult to see how to analyze general relativity according to these ideas)

Maxwell fields are classical systems too

free field

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{B} &= 0\\ \dot{\mathbf{B}} &= -\boldsymbol{\nabla}\times\mathbf{E}\\ \boldsymbol{\nabla}\cdot\mathbf{E} &= 0\\ \dot{\mathbf{E}} &= \boldsymbol{\nabla}\times\mathbf{B}\end{aligned}$$

components of dynamical vector field: $(0,\, -\boldsymbol{\nabla}\times\mathbf{E},\, 0,\, \boldsymbol{\nabla}\times\mathbf{B})$

turn on interaction

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{B} &= 0\\ \dot{\mathbf{B}} &= -\boldsymbol{\nabla}\times\mathbf{E}\\ \boldsymbol{\nabla}\cdot\mathbf{E} &= \rho\\ \dot{\mathbf{E}} &= \boldsymbol{\nabla}\times\mathbf{B} + \boldsymbol{\jmath}\end{aligned}$$

dynamical vector field: $(0,\, -\boldsymbol{\nabla}\times\mathbf{E},\, \rho,\, \boldsymbol{\nabla}\times\mathbf{B} + \boldsymbol{\jmath})$
interaction vector field: $(0,\, \mathbf{0},\, \rho,\, \boldsymbol{\jmath})$

(why $\mathbf{B}$ is "configuration" in Lagrangian formulation of Maxwell theory, and $\mathbf{E}$ "velocity")

Even classical fields not in the Newtonian framework have the same structure!

in classical mechanics, evolution characterizes time and its geometry, while interaction (including "isolation") characterizes space and its geometry

both jointly determine the full 4-dimensional flat affine geometry of classical ("neo-Newtonian") spacetime

1. because the family of dynamical vector fields has the structure of an affine space, the dynamical evolutions (integral curves) have a natural affine parametrization
2. that determines the temporal metric: the ratios of the "lengths" of different intervals on the same curve is determined by the affine structure
3. the interactions determine configuration space; how to show that it encodes Euclidean $\mathbb{R}^3$?
4. at every point of configuration space, the tangent plane can be naturally identified with the free dynamical vector field (the system moving freely with all possible velocities with respect to the frame of reference naturally defined by that configuration, *viz*., the stationarity of that configuration)
5. those velocity differences correspond to Galilean boosts, the family of which naturally has the structure of Euclidean $\mathbb{R}^3$, once the zero point is fixed, which the frame of reference does

   (it begs no question to use Galilean boosts: I am not assuming they are symmetries of the system; I need only that interactions, considered as differences between dynamical evolutions, are invariant under the addition of boosts to the dynamical evolutions—they are symmetries of *interactions*)

6. the free vector field on the Newtonian space of states induces an affine connection on the induced configuration space: at each point of configuration space, each tangent vector ("velocity") determines a unique geodesic, that one that lifts to one of the integral curves of the free vector field on the space of states

7. this affine connection is flat

8. it projects down to a flat affine connection on the Euclidean $\mathbb{R}^3$ constructed previously, and is naturally identified with the canonical flat affine connection on Newtonian 3-space

9. Lie-deriving it with respect to the free evolution vector field yields the canonical flat affine connection on 4-dimensional Newtonian spacetime

the distinction between time and space is built in to the structure of the dynamics, as is the geometry of spacetime itself

Theorem (Curiel 2019)

The Euclidean geometry of ordinary space, the metrical structure of time, and the flat affine geometry of 4-dimensional Newtonian spacetime is entirely determined by the dynamical structure of a classical system.

Conjecture

All the affine structures induced on configuration space by a different choice of conservative second-order vector field are equivalent in the sense that they yield the same Newton-Cartan spacetime (i.e., in the same equivalence class in the sense of Trautman, related to each other by addition of an appropriate difference tensor to the derivative operator—see Malament 2012)).

Hamiltonian mechanics

- phase space is a symplectic manifold
- family of dynamical vector fields $\mathfrak{D}$ is a Lie algebra, formed from solving Hamilton's equation for all Hamiltonians ($\boldsymbol{\xi} = \boldsymbol{\Omega}(\boldsymbol{\nabla} H)$)
- only fixed relations between $\mathbf{q}$ and $\mathbf{p}$ are canonical Poisson brackets:

$$\{q_i,\, q_j\} = 0$$
$$\{q_i,\, p_j\} = \delta_{ij}$$
$$\{p_i,\, p_j\} = 0$$

- "interaction" is adding another Hamiltonian:

$\mathfrak{D}$ is same as $\mathfrak{I}$

Theorem (Curiel 2014)

Fix an even-dimensional, orientable manifold with a Poisson bracket structure and a vector space of vector fields on it. Then:

1. *the Poisson bracket arises from a symplectic structure,*
2. *and the vector space includes all and only solutions to Hamilton's equation formulated with it*

if and only if

1. *the vector fields span the tangent planes,*
2. *and the manifold has a family of coordinate systems whose coordinate functions satisfy the canonical Poisson bracket relations,*
3. *and the associated coordinate vector fields leave the vector space invariant under the action of the Lie bracket,*
4. *and the vector space is maximal under these properties.*

Hamiltonian mechanics is not a classical framework:

1. the families of evolution and interaction vector fields are identical, having the structure of a Lie algebra (vector space) – "interaction" is just adding another Hamiltonian
2. there is no principled distinction between "configuration" and "momentum"; in particular the latter is not the dynamical derivative of the former, and their fixed relations are symmetric
3. *a fortiori*, there is no naturally distinguished "free evolution" vector field (the zero vector field is not a good candidate—too degenerate—one wants to allow "constant changes to configuration" for free evolution)

if I give you a manifold and tell you how to formulate Hamilton's equation on it (a symplectic structure), *i.e.*, if I tell you the structure of the dynamics:

1. you can't tell me what's configuration and what's momentum ("space" versus temporal derivatives)
2. you can't distinguish interactions from evolutions ("interventions")
3. you can't tell what is free evolution ("isolation")
4. you can't reconstruct structure of spacetime, how dynamics hooks up to it

BUT once I fix all this (by fiat, by divine revelation, . . .), then there is clean separation between a system's degrees of freedom and those of its environment: I can ignore details of environment's degrees of freedom, treat it as black-box, everything relevant encoded in "interaction Hamiltonian"

Why, then, does Hamiltonian mechanics work so well to model "classical" systems? Because

1. by fiat, we identify some variables as "momentum"
2. then restrict attention only to Hamiltonians having a special form, those "purely quadratic in momentum"
3. this reproduces the fixed kinematical relations between configuration and velocity in classical systems

examples:

- $H = \frac{1}{2}p^2 + \frac{1}{2}q^2$ is a simple harmonic oscillator, yielding the correct classical relation $v = \dot{q}$ under the Legendre transform
- think of $H = \frac{1}{2}p^2 + \frac{1}{2}q^2 + p$, however—that yields $v = \dot{q} + 1$, which is physically meaningless
- $H = p^{5/2}$ or $H = \sin(q)$ are even worse

Hamiltonian mechanics is an odd no-man's land, sharing features of both classical and quantum mechanics:

1. there is no mathematical or conceptual distinction between interaction and evolution ($\mathfrak{I} = \mathfrak{D}$);
2. *but* there is still a clean separation of the system's degrees of freedom from those of the environment, in the precise sense that I can "turn on" an interaction (in scare-quotes, since this is not distinct from an evolution) while treating the environment as a black-box, *i.e.*, not needing to give an explicit representation of its degrees of freedom and their evolution in order to fully treat the character and evolution of the system.

One way to underscore the point:

1. in so far as the genus of a system is largely determined by its set of kinematic and dynamic quantities, turning on an interaction does not change its genus.
2. A Navier-Stokes fluid is a Navier-Stokes fluid in large part because it bears the physical quantities shear viscosity, bulk viscosity, thermoconductivity, fluid velocity, shear-stress, heat flux, *et al*.
3. the first three are treated as constants, and fluid velocity is a vector, shear-stress a symmetric tensor, and so on
4. turning on an interaction does not change any of this, and therefore does not change the type of system one is dealing with

So perhaps Hamiltonian mechanics is best described as quasi-classical...

a *criollo* miscegenation between classical and quantum mechanics, in which

- one loses the distinction between interaction and evolution
- and the intrinsic differentiation between configuration and momentum
- but one still has a clean separation of the system's degrees of freedom from those of the environment.

1. dynamical vector fields ("evolutions") are unitary flows on Hilbert space
2. unitary flows are exponentiated Hamiltonians (self-adjoint operators)
3. "interaction" is just adding another Hamiltonian
4. only fixed relation between "configuration" and "momentum" is canonical commutation relation

$$[Q,\, P] = i\hbar I$$

if I give you a Hilbert space and its family of self-adjoint operators, *i.e.*, the dynamics:

1. you can't distinguish interactions from evolutions ("external interventions" such as coupling the system with another)
2. you can't tell me what's configuration and what's momentum
3. you can't tell what is free evolution ("isolation")
4. indeed, even if I give you a "standard" Hamiltonian in some funky basis, and ask you to decompose it into its "free" part and its "interaction" part, you won't be able to do it

I need an explicit representation both of degrees of freedom of the environment, and their intertwining with those of the system, in order to appropriately and adequately treat the system "during an interaction".

More specifically, I need:

1. *to change the space of states I use to treat the system*, now the tensor product of its "isolated" Hilbert space with that of the environment;
2. to change its set of physical quantities, now the algebra of observables on the tensor-product Hilbert space;
3. to change the structure of the quantities encoding evolution, now Hamiltonians on the tensor-product.

that most quantum of quantum mechanical phenomena has now made its entrance, *entanglement*

$\Longrightarrow$ no clean separation in quantum mechanics between a system's degrees of freedom and those of its environment

if I want to understand how a system evolves under interaction with the environment, I have to know how to model the environment's degrees of freedom and dynamics

(akin to Don Howard's reconstruction of Bohr's understanding of entanglement, and his insistence on "complementarity" in quantum measurements, made rigorous) in quantum mechanics, the transition to non-classicality is complete

How do we distinguish the Qs and the Ps in quantum mechanics, and so define a "free" evolution? Two ways:

1. by fiat
2. introduce representation of Poincaré or Galileian group *à la* Wigner, and define Q as the generator of momentum translations, *etc*.

In both cases, we must explicitly and by hand hook the dynamics up to the background spacetime structure to get a principled distinction between configuration and momentum, and so define "free" and "interacting"

we don't get it from the dynamics alone
as in classical mechanics

WILD SPECULATION

perhaps the lesson is that there is at bottom no real difference between Q and P in quantum mechanics

this would strongly suggest that quantum gravity should not be formulated in a 3+1 framework

Perhaps most strikingly, moreover, in so far as the genus of system is determined in part by its set of kinematic and dynamic quantities, turning on an interaction *changes its genus*, in so far as it now has a new set of both kinds of quantities—

we are now working, in a precise sense, with an entirely new kind of physical system!

Or, (a little) more precisely:

- the system *cum* environment now has a distinctive set of kinematic and dynamic quantities, which cannot be determined as or reduced to a function of those of the isolated system and those of the isolated environment taken individually
- and one must take account of those new "global" quantities in order to give an appropriate and adequate treatment of the original system

Interaction and Evolution in Newtonian and Lagrangian Mechanics

Classical Spacetime Structure

Interaction and Evolution in Hamiltonian Mechanics

Interaction and Evolution in Quantum Mechanics

Indefinite Causal Orders

[DRAW SOME PICTURES]