Energy, Entropy, and the Intimate Relations between the Two in Semi-Classical Gravity

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Remembrance of Things Past—Maxwell's Way: Energy and Entropy in Classical Thermodynamics

In the Shadow of Old Energy and Entropy Conditions Blooming in GR

Wall's Way—Sodom and Gomorrah

as a philosopher and gad-fly one of my duties is to take a step back and remind us all from time to time of some basic ideas that we tend to lose sight of, so much part of the ground of our framing conceptual landscape are they

so I begin with a few basic—almost trite—observations about energy and entropy in classical physics (non-relativistic, non-quantum), to set the stage for the discussion of more advanced, abstract matters, far removed from our empirical access

in any event, one never goes wrong by invoking Fermi's Pleasure Principle to begin a talk what energy and entropy share in common in classical physics:

1. their universality—every theory treating every kind of physical system attributes (or can attribute) both to them, except perhaps entropy for the simple mechanics of rigid bodies

2. and except—*most interestingly*—the Newtonian gravitational field (if one tries to attribute a "Bekenstein entropy" to a "Newtonian black hole", it doesn't work for many reasons, among them that the geometry of Newtonian spacetime does not admit event horizons, nor even trapping horizons)

3. their fungibility—all forms of each are, respectively, transformable into each other

4. both are affine quantities (no natural zero point)

5. each obeys a characteristic relation among all their possible forms, for energy an equality (conservation) and for entropy an inequality (the Second Law), and both of those relations have an intimate connection with time

what energy and entropy share in common in classical physics (cont.):

6. indeed, there are (we tend to forget) two entirely separate conceptions of energy in classical physics, the generator of time translations ("Hamiltonian") and the capacity to do work ("Lagrangian")
7. that they are *a priori* independent of each other is seen most clearly by the fact that, since work depends only on spatial paths, the latter makes not even an implicit reference to time[1]

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1. the work 1-form in Lagrangian mechanics is transverse to "accelerations"; the energy, as a function of the Lagrangian, does not generate dynamical evolution as the Hamiltonian does; one cannot represent "work" at all, much less dissipative processes, in Hamiltonian mechanics

what energy and entropy share in common in classical physics (cont.):

8. the two conceptions of energy are brought into contact with each other through the principle of conservation, which itself is balanced by the introduction of the idea of *heat* (neither a generator of time translations nor the capacity to do work) and thus, inevitably, the introduction of entropy from energetic considerations
9. thus there are in fact three conceptions of energy in classical physics, united by the conservation principle and by the Second Law (for the latter, most explicitly in the form of the Kelvin Postulate)
10. and, tantalizingly, by the characterization of equilibrium, as the maximizer of entropy and the minimizer of free energy (Kelvin's "Principle of Dissipated Energy")

Now when the appearance of one thing is strictly connected with the appearance of another, so that the amount which exists of the one thing depends on and can be calculated from the amount of the other which has disappeared, we conclude that the one has been formed at the expense of the other, and that they are both forms of the same thing.

– James Clerk Maxwell
Theory of Heat (1891, ch. IV, p. 93)

note how careful Maxwell's formulation is: it applies *both* to energy (conservation, an equality) *and* to entropy (Second Law, an inequality) nonetheless, there are clear conceptual and physical differences, most fundamentally:

1. they are not jointly fungible
2. there is no such thing as an entropometer, but there are ergometers ("entropy mediates no physical couplings between systems"—it is a purely *modal* quantity)
3. energy conservation must be postulated, but plays essential roles in derivation of many important, general results
4. entropy increase can be derived, but plays no essential role in derivation of any important, general results ("the Second Law lacks fecundity")
5. entropy constrains the possible future transformations of a system more severely than does energy conservation

Emden (1938, p. 909) (see rest of article as well):

> *As a student, I read with advantage a small book by F. Wald entitled 'The Mistress of the World and her Shadow'. These meant energy and entropy. In the course of advancing knowledge the two seem to me to have exchanged places. In the huge manufactory of natural processes, the principle of entropy occupies the position of manager, for it dictates the manner and method of the whole business, whilst the principle of energy merely does the bookkeeping, balancing credits and debits.*

many (but not all) properties of energy carry over from classical physics, and are even deepened and become more subtle, for they are now united (along with momental quantities) in a single new quantity, *viz.*, stress-energy:

universality matter couples with "gravity" ("the universal force", *viz.*, spacetime curvature) by way of stress-energy, attributed promiscuously to every kind of matter field

fungibility every form of stress-energy can be transformed (in principle) into every other kind

conservation one now has in general only "hyperlocal" conservation ($\nabla^n T_{na} = 0$), no integral conservation laws ("gravitational energy is non-local")[2]

zero point no longer an affine quantity ("$T_{ab} = 0$" is unambiguous)

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2. Perhaps we can posit a non-local gravitational energy to save conservation, as they did heat in the 19th Century? And so gravitational energy really is an energy associated with dissipative, hidden degrees of freedom. . .

and energy conditions make their first appearance

- constraints on measure of curvature (usually Ricci or Einstein), given physical interpretation by way of stress-energy tensor in EFE
- EFE on its own has almost no physical content without them
- almost every general, deep result in GR assumes an energy condition for its proof
- by way of guaranteeing focusing of geodesics

most characteristic role of energy in GR also most characteristic of energy conditions:

- in classical GR, as in most theories, one has a great deal of freedom in what one takes as primitive and what as derived
- think of the geodesic principle and covariant conservation of stress-energy, inter-derivable
- this is not true of the classical energy conditions, neither pointilliste nor impressionist
- one *can't derive energy conditions* in classical GR
- they are always taken as primitive (Curiel 2017)
- perhaps they reach down to and get ahold of spacetime structure at a very deep level?

- this is not so of *entropy* conditions in classical GR[3]

- Bousso (1999a, 1999b), *e.g.*, used the DEC in his original work to motivate his covariant entropy bound ("the total entropy flux S_L through any null hypersurface L satisfying some natural geometrical conditions must be such that $S_L \leq A/4$, where A is a spatial area canonically associated with L")

- Flanagan et al. (2000) then proved it using the NEC

3. Putting aside for the moment what one means by entropy, here, what kind of entropy one is dealing with—just assume for the moment that "black hole area is something like an entropy".

the relations between energy and entropy are neighborly, but not intimate:

1. relation between energy conditions and entropy conditions is "one way" only
2. there is no explicit unification of different types of entropy (in something like a GSL, *e.g.*), as there is for energy
3. as in classical thermodynamics—if one accepts the orthodox dogma (Wald's Way), that there is no consistent thermodynamical theory of purely classical black holes—energy and entropy are not jointly fungible (throwing mass into a classical black hole doesn't increase its entropy)
4. there is still no entropometer
5. and relation of *both* stress-energy and entropy to equilibrium (existence of timelike Killing field) is obscure at best

in any event, already energy here goes beyond the role it plays in non-relativistic physics. . .

to paraphrase Emden's marvelous remark, in the huge manufactory of natural processes, energy begins to occupy the position of, if not manager, at least assistant manager, for it constrains the manner and method of the whole business

Carlo Rovelli (personal communication, his emphases):

> *Entropy (and thermodynamics in general) is not yet understood (by anybody) even in CLASSICAL gravity. . . . What is well understood (since Tolman) is the thermodynamics of matter fields on a given geometry, which is to say: disregarding dissipation into gravitational degrees of freedom. But dissipation into gravitational degrees of freedom has no reason not to occur (which in the Black Hole case is the classical analog of your observation about gravity getting entangled with the Hawking radiation, for instance). So the classical thermodynamics of the gravitational field is a topic for which we do not have a science at all, yet.*

I agree

Maybe we can make some progress in SCG.

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Wall's Way—Sodom and Gomorrah

How did classical general relativity know that the horizon area would turn out to be a form of entropy, and that surface gravity is a temperature?

Ted Jacobson
"Thermodynamics of Spacetime:
The Einstein Equation of State"

first hints of more intimate relations between energy and entropy,[4] provisionally accepting BHT:

fungibility energy can now be directly transformed into entropy ("throw stuff into black hole"), and vice-versa (Hawking radiation); each is a direct measure of the other ("area and mass tell you each other")

zero point they both have natural zero points, which is the same state (Schwarzschild $M = 0$)

equilibrium heuristic but compelling arguments that stationary black holes minimize free energy ("$M - M_{\text{irr}}$") and maximize entropy

entropometer! we can measure area of event horizon directly (not a modal quantity)—which is also, in this case an ergometer, as area gives you mass

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4. Here, we take the black hole area to be uncontroversially (more or less) a thermodynamical entropy.

but the relations—including equalities—between entropy and energy even beyond those just stated become now a promiscuous, libertine, orgiastic debauch...

Sodom and Gomorrah, during the fun times!

- one of the most natural ways to think of gravitational energy in GR (yet largely unexplored) is by way of geodesic deviation (focusing) (Penrose 1966; Curiel 1997)
- one can extract energy from a gravitational field when and only when there is geodesic deviation (focusing) (Bondi and McCrea 1960; Curiel 1997)
- as we'll see, entropy now becomes associated with geodesic deviation (focusing) as well

entropy conditions[5] take on the classical role of energy conditions, by guaranteeing geodesic focusing:

1. GSL proves a singularity theorem, and rules out traversable wormholes, negative masses, other forms of faster-than-light travel between asymptotic regions, restarting inflation and CTCs: Wall (2013)
2. Bousso bound proves a singularity theorem: Bousso and Shahbazi-Moghaddam (2022)
3. quantum Penrose inequality using generalized entropy of bulk light sheets to constrain lower bound of ADM mass: Bousso et al. (2019)
4. quantum focusing implies singularity theorems, the GSL and boundary causality: Shahbazi-Moghaddam (2022)

(*N.b.*: the last only a conjecture, with supporting plausibility arguments and evidence from test cases)

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5. It now becomes almost wholly unclear what is meant by entropy in any given application, and whether, in any event, they are jointly consistent.

the principle of entropy increase (GSL) becomes fecund!

one now has, for the first time, not only derivations of energy conditions, but ones *based on entropy conditions*, and vice-versa

1. proving the (A)ANEC from the GSL: Wall (2010)
2. proving the ANEC from the QNEC: Bousso et al. (2016)
3. proving the NEC from the GSL: Parikh and Svesko (2017)

(we'll come to the QNEC—Quantum Null Energy Condition—next)

QNEC, Bousso et al. (2016)

1. any point p and null vector k^a define (at least locally) a null plane N
2. given any codimension-2 surface Σ that contains p and lies on N, consider the von Neumann entropy S_{out} of the quantum state of the ambient quantum fields restricted to one side of Σ
3. a second variation S''_{out} can be defined by deforming Σ along N, in a small neighborhood of p, by an area $\mathcal{A}$
4. QNEC:
$$\langle T_{kk}(p)\rangle \geq \frac{\hbar}{2\pi} \lim_{\mathcal{A}\to 0} S''_{\text{out}}/\mathcal{A}$$

Figure

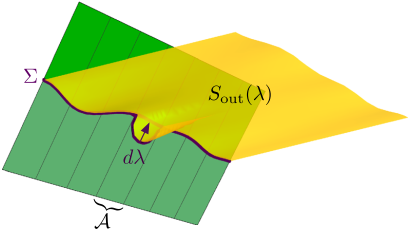

FIG. 1. The spatial surface Σ splits a Cauchy surface, one side of which is shown in yellow. The generalized entropy $S_{\rm gen}$ is the area of Σ plus the von Neumann entropy $S_{\rm out}$ of the yellow region. The quantum expansion Θ at one point of Σ is the rate at which $S_{\rm gen}$ changes under a small variation $d\lambda$ of Σ, per cross-sectional area $\mathcal{A}$ of the variation. The quantum focusing conjecture states that the quantum expansion cannot increase under a second variation in the same direction. If the classical expansion and shear vanish (as they do for the green null surface in the figure), the quantum null energy condition is implied as a limiting case.

[*** shamelessly cribbed from Bousso et al. (2016) ***]

one gets equivalences of entropic and energetic quantities:

1. Leichenauer et al. (2018): for null shape deformations as they appear in the QNEC, modulo a plausible, supported conjecture, second variations of the von Neumann entropy determine the full stress-energy tensor expectation value as an equality (and so, *à la* Jacobson 1995, one gets the EFE)
2. Wang (2020): (quasi-local) Bartnik-Bray inner mass exactly equals the (generally non-local) irreducible mass corresponding to the (generally non-local) outer entropy (Engelhardt and Wall 2018)

perhaps most striking (argument due to Manus Visser):

1. set
$$\langle K_\xi \rangle := \int_\Sigma \langle T_m{}^n \rangle \xi^m \mathsf{d}\Sigma_n$$
2. then SCG First Law:
$$\delta M = \frac{\kappa}{8\pi} \delta A + \delta \langle K_\xi \rangle$$
3. invoke First Law of quantum thermodynamics, *a.k.a.*, First Law of entanglement
$$\delta \langle K_\xi \rangle = T_{\mathsf{H}} \delta S_{\mathsf{ent}}$$
where $\delta S_{\mathsf{ent}} = -\operatorname{Tr} \rho \log \rho$ and $\rho = \dfrac{1}{Z} e^{-\beta_H K_\xi}$
4. $\Rightarrow \delta M = T_{\mathsf{H}} \delta S_{\mathsf{gen}}$
where $S_{\mathsf{gen}} = A + S_{\mathsf{ent}} = S_{\mathsf{BH}} + S_{\mathsf{ent}}$
and $T_{\mathsf{H}} = \dfrac{\kappa}{8\pi}$

S_{gen} obeys *both* a First Law and a Second Law!!!

Now when the appearance of one thing is strictly connected with the appearance of another, so that the amount which exists of the one thing depends on and can be calculated from the amount of the other which has disappeared, we conclude that the one has been formed at the expense of the other, and that they are both forms of the same thing.

– James Clerk Maxwell
The Theory of Heat (ch. IV, p. 93)

Are energy and entropy different aspects, different forms, of the same underlying entity? Should this be one of the unifications we seek now in physics?

two paths to the EFE: (1) entropy

further hints from Jacobson (1995):

1. assume form of entropy (proportional to area)
2. assume form of heat (matter flux of particular sort, $= 0$ when $T_{ab} = 0$)
3. assume Clausius relation (temperature from Unruh effect)
4. $\Rightarrow$ Einstein field equation as consistency condition on thermodynamical relations ("equation of state")

$\Rightarrow$ *null energy condition as local law of entropy increase*

(see also Curiel et al. 2020)

two paths to the EFE: (2) stress-energy

Theorem (Curiel 2019)

The only two covariant-index, symmetric, divergence-free, second-order concomitants of the metric with physical dimension of stress-energy (in geometrized units) are constant multiples of the Einstein tensor.

$\Rightarrow$ *physical dimension of stress-energy (T_{ab} invariant under constant recalings of g_{ab}) determines coupling of curvature to matter*

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