Translation of Talks from the 1955 Bern Conference "Fünfzig Jahre Relativitätstheorie"

Yvonne Fourès-Bruhat, « Le probléme de Cauchy dans la théorie relativiste de l'électromagnétisme et dans la théorie unitaire de Jordan-Thiry »

André Lichnerowicz, « Problèmes généraux d'intégration des équations de la relativité »

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Translator's Notes

Translator's footnotes are marked by numbers preceded by 'T', *e.g.*, 'T1'. All other footnotes are the author's. They are Old Skool, marked by a superscripted number prepended to a right parenthesis, *e.g.*, '[1])'.

Y. Fourès-Bruhat, « Le probléme de Cauchy dans la théorie relativiste de l'électromagnétisme et dans la théorie unitaire de Jordan-Thiry »

The Cauchy Problem in the Relativistic Theory of Electromagnetism and in the Unitary Theory of Jordan-Thiry

by Mme. Y. Fourès-Bruhat (Aix-Marseille)

1. In general relativity the electromagnetic field, external form (cf. [1]) $F = F_{ij} dx^i \wedge dx^j$, and the metric, quadratic form $g = g_{ij} dx^i dx^j$, are linked by the equations of Einstein

$$S_{ij} = R_{ij} - \frac{1}{2} g_{ij} R = \xi T_{ij} \tag{1}$$

$$T_{ij} = \frac{1}{4} g_{ij} F_{hk} F^{hk} - F_{ih} F^j_j$$

and the equations of Maxwell which can be written (for a zero current-vector):

$$dF = \delta F = 0 \tag{2}$$

where dF and δF designate differentiation and codifferentiation with respect to the metric g.[T1]

I first look for a solution such that $F = d\phi$. Equation (2) takes then the form

$$\delta d\phi. \tag{2'}$$

Fix, at the initial instant $x^4 = 0$, ϕ and g and the derivatives $\partial\phi_j/\partial x^4$, $\partial g_{ij}/\partial x^4$ satisfying the necessary conditions

$$S^4_i = 0, \quad (\delta d\phi)^4 = 0. \tag{3}$$

The initial coordinates being isothermal (cf. [2]),[T2]

$$F^i \equiv \nabla_j g^{(i)j} \equiv g^{jh} \Gamma^i_{jh} = 0 \quad \text{for } x^4 = 0 \tag{4}$$

and the potential vector ϕ being normalized by

$$\delta\phi = -\nabla_i \phi^i = 0 \quad \text{for } x^4 = 0. \tag{5}$$

T1. 'Differentiation' here clearly means "differentiation with respect to the exterior derivative" (and so independent of the metric), and 'codifferentiation' means "divergence with respect to the Levi-Civita derivative operator associated with the metric".

T2. These are perhaps better known as harmonic coordinates.

We deduce from (3), (4) and (5)

$$\partial F^i/\partial x^4 = \partial(\nabla_i \phi^i)/\partial x^4 \quad \text{for } x^4 = 0.$$

The identities

$$R_{ij} \equiv \frac{1}{2} g^{hk} \frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} + H_{ij}(\partial_h g_{lk},\, g_{lk}) + \frac{1}{2} g_{ih} \partial_j F^h + \frac{1}{2} g_{jh} \partial_i F^h \tag{6}$$

$$(\delta\partial\phi)_i \equiv g^{hk} \frac{\partial^2 \phi_i}{\partial x^h \partial x^k} + P_i(\partial_k \phi_h,\, \partial_k g_{lh}, g_{lh}) + \partial_i(\nabla_h \phi^h + F^h \phi_h) \tag{7}$$

show that in isothermal coordinates, and for a potential vector normalised by $\delta\phi = 0$, the equations of MAXWELL-EINSTEIN take the form of a system of second-order hyperbolic, nonlinear equations where the derivatives of second order are separated[T3] and have the same coefficients for all equations. I have constructed [3] a (unique) solution of the Cauchy problem for such a system without any assumption of analyticity: its value at a point depends only on the initial data inside a conoid the vertex of which is the point (from which follows wave propagation and the identity of the gravitational and electromagnetic propagations) and depends continuously on the initial data.[T4]

The conservation identities and the identity $\delta\delta\psi = 0$ allow the demonstration that this solution is isothermal and that ϕ satisfies the condition $\delta\psi = 0$. We have thus effectively constructed a solution of the MAXWELL-EINSTEIN equations (1) and (2). We [also] show that this solution is physically unique.

T3. *I.e.*, they do not appear in terms multiplied by the field values or their first derivatives.

T4. She elliptically draws attention to a nexus of subtle and important points all grounded in the fact that she has not assumed the solution to be an analytic function (as, indeed, the solutions to hyperbolic partial-differential equations in general are not, as opposed to those of elliptic and parabolic partial-differential equations). That implies, first, that traditional brute force analytical techniques for proving existence and uniqueness—*e.g.*, calculating coefficients term by term in a Taylor-series expansion—were not available to her. She had to develop more manifestly "geometrical" methods to solve the problem. Second, her solution is more robustly realistic as a solution to a problem in *physics* (as opposed to pure mathematics). Analytic functions are physically pathological in so far as their values *everywhere* are determined by their values in arbitrarily small regions *anywhere*, a peculiar form of hyperdeterminism. To know an analytic solution in an arbitrarily small region, therefore, determines its initial data everywhere, and in particular in regions outside the causal past of the region. The values of her solution in an arbitrarily small region, by contrast, provably depend only on initial data contained in the causal past of that region.

Alex Blum (email, 16 Nov 2021):

> I'm a little confused by one point in the Fourès-Bruhat paper: in the older papers I've read (going back to Hadamard) the focus always seems to be on the analyticity of the initial value data, while she emphasizes the assumed analyticity of the putative solution. Is there a subtle distinction there?

I reply:

> There is the fact that that hyperbolic PDEs are the only ones whose solutions are not guaranteed to be analytic (as opposed to solutions of parabolic and elliptic), irrespective of the character of the initial data. But that doesn't seem like it would answer your question, since, presumably, everyone reading her paper and listening to her presentation would already know that. The only thing that really makes sense to me is the interpretive issue I point to, about causality, which really does turn on the difference between the analyticity of initial data restricted to a 3-surface (who cares about "hyper-determinism" there, it's no worse than the kinds of global correlations imposed on a spacelike 3-surface by the fact that, *e.g.*, the divergence of the electric field here is determined by a local charge distribution way over there), and that of a 4-d solution, which seems manifestly problematic (kind of like Bell correlations, really).

We can equally as well determine F by $\Box F \equiv d\delta F + \delta dF = 0$ without using the vector potential ϕ whose existence is assumed by the previous reasoning (ϕ could exist only locally). A uniqueness theorem then guarantess the existence of ϕ given its existence at the initial time.

2. *Unitary theory of* JORDAN-THIRY: the fifteen unknowns $\gamma_{\lambda\mu}$ are the coefficients of the metric of a five dimensional Riemannian space, cylindrical with respect to x^0:

$$ds^2 = \gamma_{\lambda\mu} dx^l dx^\mu = -\xi^2(dx^0 + \beta\phi_i dx^i) + d\hat{s}^2$$

$d\hat{s}^2 = g_{ij}dx^i dx^j$ is the metric of the space-time, ϕ_i the vector potential, ξ a fifteenth potential whose interpretation is up for discussion (cf. the talk by A. LICHNEROWICZ).[T5]

Outside the mass distribution, the unknowns $\gamma_{\lambda\mu}$ satisfy the equations:

$$R_{ab} = 0 \tag{8}$$

(R_{ab}, the Ricci tensor of the five-dimensional space). We give at the initial time $x^4 = 0$ the $\gamma_{\lambda\mu}$, $\partial\gamma_{\lambda\mu}/\partial x^4$ (*i.e.*, g_{ij}, ϕ_i, ξ and their first derivatives) satisfying the necessary conditions:

$$S^4_\lambda = 0 \quad \text{for } x^4 = 0. \tag{9}$$

The initial coordinates being isothermal

$$F^\mu \equiv \nabla_\lambda \gamma^{\lambda(\mu)} = 0 \quad \text{for } x^4 = 0, \tag{9}$$

we derive from (9) and (10)

$$\partial_f F^\mu = 0 \quad \text{for } x^4 = 0. \tag{10}$$

A decomposition analogous to (6) allows us to solve equations (8) in isothermal coordinates, the conservation conditions $\nabla_\lambda S^\lambda_\mu \equiv 0$ showing again that we have effectively a physically unique solution of these equations. This solution depends continuously on the initial data (in particular ξ remains constant if it is so at the initial time).

If we make $\xi = \text{const.}$ in the first fourteen equations of the unitary theory of JORDAN-THIRY, we obtain the equations of the KALUZA-KLEIN theory, equivalent to the MAXWELL-EINSTEIN equations, which allows us to recover the results of §1.

Bibliography

[1] LICHNÉREOWICZ, A., *Théories relativistes de la gravitation et de l'électromagnétisme*, Masson (1955).

[2] DARMOIS, G., *Equations de la gravitation einsteinienne*, Memorial Sci. Math. (1927).[T6]

[3] FOURÈS-BRUHUT, Y., Acta Matem. (1952).[T7]

T5. Is she being catty or serious? I suspect the former.

T6. The full reference is

Georges Darmois, 1927, *Les équations de la gravitation einsteinienne*, Mémorial des Sciences mathématiques dirigé par Henri Villat, fasc. XXV, Gauthier-Villars et C^{te}, Paris.

T7. The full reference is

Y. Fourès (Choquet)-Bruhut, 1952, "Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires", *Acta Mathematica* 88:141–225, doi:10.1007/BF02392131

A. Lichnerowicz, « Problèmes généraux d'intégration des équations de la relativité »

General Problems in Integrating the Equations of Relativity[T1]

by André Lichnerowicz (Collège de France)

In this lecture, I intend to show what are the problems posed by the mathematical study of the relativistic equations of gravitation and electromagnetism and to indicate the main results obtained in this field during the most recent years. This lecture will be devoted mainly to 'classical' general relativity, but, along the way, I will be led to show that the mathematical problems posed by unitary theories, whether they are of the Jordan-Thiry type or non-symmetric type, do not differ much from those concerning general relativity. As we shall see, the few mathematical facts brought to light by these theories help shed some light on the fundamental difficulty of unitary theories: to obtain a precise physical interpretation of the elements of the refined geometrical schemes they bring into play.

I will add that the spirit of my exposition will be that of the mathematical physicist.

1. The Structure of the Field Equations

1. The space-time manifold.

In any gravitational field theory, the primitive element is constituted by a 4-dimensional "space-time" manifold, endowed with the structure of a differentiable manifold, which it is necessary to fix in what follows.

For reasons closely related to the covariance of the formalism, and that will appear in detail through the analysis of the equations of the gravitational field, we are led to suppose that in the intersection of the domains of two admissible coordinate systems, the local coordinates of a point in one of the systems are 4-times differentiable functions of the coordinates of this point in the other system, the first and second derivatives being continuous, the third and fourth derivatives being only piecewise continuous, and the Jacobian non-zero.[T2]

We will render this by saying that the manifold V_4 is (C^2, piecewise C^4).

A Riemannian metric ds^2 of normal hyperbolic type[T3] is defined on V_4, with one positive and three negative squares.[T4] The local expression of this metric in an admissible coordinate system is:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \quad (\alpha,\ \beta \text{ and all Greek indices} = 0,\ 1,\ 2,\ 3). \tag{1}$$

T1. Early on in the translating, I got tired of transcribing the debauch of indices in the mathematical formulæ, and so began inserting images of the formulæ taken directly from the original (Lichnerowicz 1956) wherever possible.

T2. This is more cautious than the standard use of smooth structures by theoretical physicists and by philosophers today, without the mania shown by contemporary mathematical physicists for reducing the order of differentiability of all possible structures by as much as possible in all standard results, come Hell or High Water and with no regard for physical significance.

T3. In modern terms, a Lorentz metric.

T4. Using this signature, rather than $(-, +, +, +)$, was, to the best of my knowledge, unusual at that time in the context of physics; it is a modern style.

The "gravitational tensor" $g_{\alpha\beta}$ is assumed exactly (C^1, piecewise C^3), which is strictly compatible with the structure imposed on V_4. Any additional global precision of the differential structure or metric, with regard to differentiability, must be regarded as physically meaningless.[T5]

The equation $ds^2 = 0$ defines at each point x of V_4 the fundamental cone at x. Its interior and exterior define respectively the timelike and the spacelike directions.[T6] For a hypersurface Σ, defined locally by $f(x^a) = 0$, to be spacelike it is necessary and sufficient that

$$\Delta_1 f = g^{\alpha\beta}\,\partial_\alpha f\,\partial_\beta f > 0 \qquad \left(\partial_\alpha = \frac{\partial}{\partial x^\alpha}\right). \tag{2}$$

If the timelike line L is represented by x^i = const. (i and all Latin indices = 1, 2, 3), we have $g_{00} > 0$, and the quadratic forms of coefficients, dual to each other,

$$g^*_{ij} = g_{ij} - \frac{g_{0i}\,g_{0j}}{g_{00}} \qquad g^{*ij} = g^{ij} \tag{3}$$

are defined to be negative.

The manifold V_4 is not topologically arbitrary since it admits a metric field of normal hyperbolic type;[T7] by using an elliptic metric[T8] we see that V_4 certainly admits a timelike direction field.[T9] The orbits of this field provide *a global system of "time lines"*.

When topological considerations are necessary, it is often admitted, more or less explicitly, that V_4 is the topological product of a 3-dimensional manifold V_3 by a 1-dimensional manifold, the 1-dimensional submanifold factors being timelike in V_4. In this case, for many problems, only the topology of V_3 is important. The usual cases are those where V_3 is homeomorphic to the ordinary $\mathbb{R}^3$ or is a compact manifold.

The different assumptions imposed on the metric (1) characterize the so-called *regular* metrics.

2. Einstein*'s equations of general relativity.*

In the following, I will designate by $R_{\alpha\beta}$ the Ricci tensor of the metric (1) and will set

T5. An explicit show of greater sensitivity than shown by most mathematical physicists today to the issue of physical significance, with regard to the order of differentiability of standard results.

T6. I here permit myself the anachronistic terms 'timelike' and 'spacelike', as the more literal translations of Lichnerowicz's 'l'orientation dans le temps' and 'l'orientation dans l'espace' might confuse the modern reader, since the English word 'orientation' now has a different meaning in the context of contemporary differential geometry.

T7. I am impressed, and somewhat confused. It is a corollary of a famous theorem due to Geroch (1969) that a manifold admits a Lorentz metric iff it is non-compact and paracompact, or else it is compact and has vanishing Euler characteristic. (Thus, for instance, neither the 2-sphere nor the Cartesian product of the long line with itself admits a Lorentz metric.) I find it hard to credit that Lichnerowicz knew anything like this result. I am therefore curious what was known at the time about the constraints topology places on Lorentzian geometry. Does anyone know a good historian of early 20th Century topology and differential geometry? (I should check Steenrod 1951.)

T8. In modern terms, a positive-definite metric.

T9. I am somewhat confident that Lichnerowicz really means "direction field" (for the definition and use of which, see, *e.g.*, Geroch and Horowitz 1979, §2) and not "vector field" for two reasons: first, it is false that every manifold that admits a Lorentz metric also admits a nowhere-vanishing timelike vector field; second, a manifold admits a direction field if and only if it admits a Lorentz metric, and the standard construction showing this works by fixing an arbitrary positive-definite metric (Geroch and Horowitz 1979, p. 219). Still, I would love to consult a historian of early 20th Century mathematics, to verify that these sorts of results were known at the time. (I should check Steenrod 1951 and Schouten (1954).)

$$S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} (R + 2\lambda) \qquad (\lambda \text{ constante cosmologique}).$$

The equations of EINSTEIN, which in the framework of general relativity limit the generality of the metric can be written:[T10]

$$S_{\alpha\beta} = \chi T_{\alpha\beta} \ . \tag{4}$$

The momentum-energy tensor $T_{\alpha\beta}$, which plays the role of source of the field, fully describes the energy state at the point considered in V_4 (the interior case) or, in the regions not occupied by energy, is identically zero (the exterior case).[T11] It thus generalizes the second term of POISSON's equation.[T12]

The tensor $S_{\alpha\beta}$,[T13] of geometrical origin, which depends only on the $g_{\alpha\beta}$ and their derivatives up to second order, is linear with respect to the second-order derivatives and satisfies the *conservation identities*

$$\nabla_\beta S^\beta_\alpha = 0 \quad (\nabla_\beta \text{ opérateur de dérivation covariante}). \tag{5}$$

The system of EINSTEIN's equations having, as we shall see, a normal hyperbolic character, the first problem we have to deal with is that of CAUCHY, which is closely related to relativistic determinism. We shall begin with a local elementary study and, in order to keep ourselves to the essentials, we shall not introduce a second term and shall consider only the external CAUCHY problem. Its preliminary study is moreover necessary for the CAUCHY problem with a second term.[T14] Our problem is therefore the following: *Problem. Given, on a hypersurface* Σ*, the potentials* $g_{\alpha\beta}$ *and their first derivatives, to determine off* Σ *the potentials supposed to satisfy* EINSTEIN*'s equations in the external case.*[T15]

On Σ, represented locally by $x^0 = 0$, the "CAUCHY data" are the values of $g_{\alpha\beta}$ and $\delta_0 g_{\alpha\beta}$. We will designate by $f(d \cdot C)$ a function whose value on Σ can be derived from the CAUCHY data by algebraic operations and derivations restricted to Σ.

T10. This is a common sentiment still today among physicists, that being a "solution to the Einstein field equation" somehow limits the possible form or character of a Lorentz metric. But that is nonsense. Every Lorentz metric is a "solution to the Einstein field equation", for that very stress-energy tensor defined by $1/8\pi$ (or $1/\chi$ for Lichnerowicz) times the Einstein tensor. It may not be a physically interesting or even meaningful stress-energy tensor, but that makes no nevermind. Generally, therefore, when people say "this metric is (not) a solution to the EFE," they usually mean, "it's (not) a metric that satisfies one of my favorite properties, such as this nifty energy condition, or being derived from a minimally coupled Lagrangian, or being an exact solution for this groovy kind of matter field, or . . . ." So what did Lichnerowicz mean?

T11. I am not familiar with the terminology 'intérieur' and 'extérieur' used in this way, but Lichnerowicz clearly defines what he means by them, so I'm using the English terms that most literally translate them, so as to remain faithful to the flavor of the original, even though it induces, at least today, non-standard terminology in English. I suspect he means 'interior' to refer to the interior of a material body (or, more generally, the interior of its 4-dimensional world-tube), and so 'exterior' to refer to the vacuum outside of such bodies. This suspicion gains credence from his later use of the terms.

T12. *I.e.*, the righthand side of Poisson's equation, the mass density in Newtonian gravitational theory or electric charge in electrostatics.

T13. Today called the 'Einstein tensor'.

T14. *I.e.*, necessary in order to treat the Cauchy problem with material sources, *i.e.*, non-zero T_{ab}. I take it Lichnerowicz really means "useful", or something like that.

T15. Conceiving of the components of g_{ab} as a "potentials", in analogy, I take it, with the scalar potential in Newtonian gravitational theory, is, I think, an old-fashioned way of understanding the problem; I think it only muddies the conceptual waters.

We will assume for the moment that Σ is spacelike ($g^{00} > 0$). If we seek to use EINSTEIN's equations to find the second derivatives $\delta_{00} g_{\alpha\beta}$ whose values on Σ are unknown, we are led to replace these equations by the equivalent system composed of the two groups of equations:

$$R_{ij} - \lambda\, g_{ij} \equiv -\frac{1}{2} g^{00}\, \delta_{00} g_{ij} + F_{ij}\,(d \cdot C) = 0 \qquad (6)$$

$$S^0_\alpha \equiv G_\alpha\,(\mathrm{d} \cdot \mathrm{C}) = 0 \qquad (7)$$

A necessary condition for the CAUCHY problem to be possible[T16] is that the equations (7) be satisfied on Σ by the CAUCHY data. On the other hand, g^{00} being $\neq 0$, the equations (6) provide the values of $\delta_{00} g_{ij}$ on Σ. No equation contains the 4 derivatives $\delta_{00} g_{\lambda 0}$ and we have to analyze this fact.

Our purely local study has been carried out in the domain of a certain coordinate system. But the data on Σ, *viz.* the CAUCHY data in the domain considered, leave open the possibility of a coordinate change preserving the numerical values of the coordinates of every point of Σ as well as the CAUCHY data.[T17] The change of coordinates

$$x^{\lambda'} = x^\lambda + \frac{(x^0)^3}{6}\left[\varphi^{(\lambda)}(x^i) + \varepsilon^{(\lambda)}\right] \qquad (\lambda' = \lambda\,\textbf{numériquement}) \qquad (8)$$

where $e^{(\lambda)}$ is infinitesimally small at the time x^0, serves the purpose. In such a coordinate change, the derivatives $\delta_{00} g_{ij}$ are not modified, while the $\delta_{00} g_{\lambda 0}$ can be given arbitrary values. By using a coordinate change where the $\phi^{(\lambda)}$ are different on either side of Σ, *which is allowed by the structure chosen for* V_4,[T18] one can make possible discontinuities of these second derivatives appear or disappear, discontinuities that are thus devoid of any physical meaning. Thus, the $\delta_{00} g_{ij}$ are continuous through Σ and the $\delta_{00} g_{\lambda 0}$ can be forced to be continuous for a suitable coordinate system.[T19]

Here we grasp the mechanism that links the covariance of the formalism to the structure chosen for V_4.[T20] This being the case, it is easy to see that the system of EINSTEIN's equations is in involution: if a ds^2 satisfies equations (6) and, *on* Σ, equations (7), then it also satisfies equations (7) off of Σ. This is an immediate consequence of the conservation identities (5). Our initial problem must be divided into two distinct problems: *Problem I or the initial conditions.* It consists in the search for CAUCHY data satisfying on Σ the system $S^0_\alpha = 0$, *viz.*, a system of initial conditions. *Problem II or the problem of evolution.* It consists in the integration of the system (6) for CAUCHY data satisfying the conditions of the first problem.

The results of this first analysis are not entirely modified if Σ is timelike.[T21] Contrarily, if Σ is tangent to the elementary cone, *i.e.*, if $g^{00} = 0$, then the second derivatives of the potentials

T16. Lichnerowicz must mean "a necessary condition for the Cauchy problem to be consistent", or perhaps even "a necessary condition for it to be possible to solve the Cauchy problem"; I am not familiar with the use of the French 'possible' for this.

T17. The original had 'S' rather than 'Σ', but that is clearly a typo. First, no region of spacetime S has been or will be defined; second, the coordinate change (8) does indeed leave the values of everything on Σ unchanged.

T18. Lichnerowicz means the fact that V_4 is the Cartesian product of $\mathbb{R}$ with $\mathbb{R}^3$ or a standard compact 3-manifold.

T19. Again, I'm digging his attitude toward questions of differentiability.

T20. This is a deep remark. He is nodding toward the fact that general covariance is a tricky issue indeed in 3+1 decompositions of spacetimes in general relativity.

T21. Gallic understatement? Or ignorance of the potential complications? See Callender (2017, ch. 8) for a thoughtful discussion of the Cauchy-like problem posed on a timelike hypersurface, and for further references to of a solution of the EINSTEIN equations may be discontinuous at the crossing of Σ; in this case, there may exist an infinite number of solutions of EINSTEIN's equations corresponding to the given CAUCHY's equations on Σ. We know classical results of the theory of partial differential equations concerning the *characteristic surfaces* (or wavefronts). C_x is the characteristic cone for the equations of EINSTEIN and the characteristic surfaces are the tangent planes to these cones; these are the solutions of

$$\Delta_1 f \equiv g^{\alpha\beta}\,\delta_\alpha f\,\delta_\beta f = 0\,.$$

We immediately deduce that the *bicharacteristics* – or rays – are the null geodesics of ds^2. Those curves that originate from a point x of V_4 are the two layers of the *characteristic conic* with vertex x.

In the case of the interior CAUCHY problem, with the form of a fluid for example, an analogous analysis can be made, but one substitutes for (7) the equations

$$S^0_\alpha = \chi T^0_\alpha$$

that relate the CAUCHY data *on* Σ with the material elements.[T22] The integration problem then concerns a system of the type (2–3) but with a second member[T23] satisfying the conservation equations

$$\nabla_\beta T^\beta_\alpha = 0.$$

Three kinds of exceptional surfaces are encountered: gravitational waves, surfaces generated by material flow lines,[T24] and hydrodynamic waves.

Within the framework of general relativity, we can always introduce the electromagnetic field, which has to satisfy the equations of MAXWELL and which makes a contribution to the second member of the equations of EINSTEIN. The analysis of the CAUCHY problem for the equations of MAXWELL shows that, in the vacuum case,[T25] C_x is still a characteristic for these equations, which

the mathematical physics literature. I was surprised to see Lichnerowicz even mention this possibility, since it is highly non-standard in the modern era to consider such problems. I suspect he does so because his seemingly Machian commitments, spelled out in §8 (starting on p. 15 below), suggest to him—somewhat naively, if so!—that the matter distribution inside 4-dimensional world-tubes ought to determine in a well posed way the metric in the vacuum regions surrounding those world-tubes, and so, in particular, the boundaries of those world-tubes being timelike hypersurfaces, one should have a well posed initial-value problem on such hypersurfaces. If it is not that, then I am truly baffled as to why he mentions it here.

T22. This depends essentially on the fact that the stress-energy tensor is Hawking-Ellis type I, *viz.*, having the same form as a perfect fluid.

T23. *Viz.*, a non-trivial righthand side.

T24. I am not entirely sure of this translation for the French 'lignes de courant'. The only translations I am already familiar with for that expression, in the context of physics, are variations on 'electric current'. He could mean the surfaces associated with electromagnetic radiation, which would make sense in context, but the start of the very next paragraph suggests he does not mean that here. If 'material flow-lines' is correct, then perhaps he means something like the paths of dust particles? The paths of the kinds of small body that were studied in the context of the problem of motion? Something like this—worldlines of material particles—is suggested by the use of the phrase in §9 (see footnote T63 below on p. 19), where it unambiguously means something like that. If it does mean "material flow-lines" in that sense here, then the "exceptional surface" is the boundary between a region of spacetime occupied by matter and the surrounding vacuum, further cementing my opinion that the essay's tacit *bête noire* is Einstein's views on matter and motion. (See my footnote T51 on p. 16 below.)

T25. I take it that he means "the absence of electric charges and currents"—and so a source-free Maxwell field—not "the absence of all matter fields", since a Maxwell field is, under any reasonable construal, a matter field. In contemporary terms, he is considering an "electrovac" solution.

establishes the identity of the propagations of the two fields; but here all the first derivatives of the electromagnetic field on a hypersurface which is not tangent to a characteristic, can be determined.

3. The equations of the unitary theory of JORDAN-THIRY[1])

In the theory of JORDAN-THIRY, the initial element consists of a differentiable manifold V_5 of class (C^2, piecewise C^4) with a metric $d\sigma^2$, which I will assume to be *hyperbolic normal*[T26] and to admit a 1-parameter group of isometries leaving no point invariant, with orbits homeomorphic to a circle and oriented $d\sigma^2 < 0$.[T27]

The manifold V_4 resulting from the quotient of V_5 by the equivalence relation defined by the group of isometries is identified with a differential space-time manifold of general relativity. From the quotient of V_5 we derive a normal hyperbolic metric $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$, an antisymmetric form $\beta F_{\alpha\beta}$ (β = const.)[T28] with zero external derivative and a scalar ξ intrinsically defined on V_4. I will deliberately not discuss for the moment the physical interpretations that one can give to this framework.[T29]

As field equations, we will adopt in V_5 equations identical to those of general relativity (4). Translated into V_4, these equations in the so-called exterior unitary case are written:

$$\left.\begin{aligned} & S_{\alpha\beta} - \frac{\beta^2 \xi^2}{2}\left[\frac{1}{2} g_{\alpha\beta} F^2 - F_\alpha^\varrho F_{\beta\varrho}\right] - \frac{1}{\xi}\left[\nabla_\alpha(\delta_\beta \xi) - g_{\alpha\beta} \Delta\xi\right] = 0 \\ & \nabla_\beta(\xi^3 F_\alpha^\beta) = 0 \\ & \frac{1}{\xi}\Delta\xi + \frac{\beta^2\xi^2}{2} F^2 = 0 \end{aligned}\right\}$$

For $\xi = 1$, the first 14 equations reduce to the form of the equations of the pure electromagnetic field in general relativity (KALUZA-KLEIN theory). The analysis of the CAUCHY problem for a hypersurface of V_5 defined by orbits of the group of isometries as well as its decomposition is similar to the previous one,[T30] the exceptional surfaces in V_4 always being the tangent surfaces to

[1]) GONSETH and JUVET have also studied, in a classical work, [trans. note: *i.e.*, as a non-quantum theory,] a 5-dimensional theory.

T26. It is clear from what follows that he uses the same signature convention as in general relativity, *viz.*, $(+, -, -, -, -)$.

T27. *I.e.*, the orbits are the integral curves of a vector field whose squared norm is everywhere negative with respect to the hyperbolic metric. (For the more mathematically persnickety: "squared norm" is mildly abusive but well intentioned shorthand for "metric inner-product with itself". I reserve the right to repeat the abuse without further remark.) Thus, the orbits are "spacelike" with respect to the 5-metric.

T28. Unfortunate notation—the β stated to be a constant here is the coefficient of the 2-form F, not its tensor index. In any event, this β being a constant, it is not clear to me why it is not absorbed into F. Perhaps it is something like a coupling constant? Because, as Lichnerowicz immediately goes on to remark, he refrains from attempting to give the mathematics a physical interaction, there are no further clues here. Perhaps someone familiar with Jordan-Thiry theory knows?

T29. The French 'volontairement' can also be translated in some contexts as 'with malice aforethought'; I believe Lichnerowicz deliberately invokes the connotation—not least, in light of the fact that the moment lasts a long time, and when it ends he opens a can of whoop-ass on Jordan and Thiry.

T30. A hypersurface so defined has a naturally induced positive-definite, *i.e.*, Riemannian, metric on it. Thus the Cauchy problem here is indeed analogous to that of general relativity: a Riemannian metric on a (d-1)-dimensional surface is treated as configuration; it and its "conjugate momentum" in the direction orthogonal the elemental cones defined by $ds^2 = 0$.[T31]

4. The equations of the non-symmetric unitary theory[2])

In the non-symmetric unitary theory, we give ourselves, on a differentiable manifold V_4, always of class (C^2, piecewise C^4),

1^0 a field of nonsymmetric tensors $g_{\alpha\beta}$ of class (C^1, piecewise C^3) with determinant $g \neq 0$ and whose associated quadratic form is normal hyperbolic,

2^0 an affine connection of class (C^0, piecewise C^2) for which we designate by S_α the torsion vector.

$R_{\alpha\beta}$ being the Ricci tensor of the connection, the field equations are based on a variational principle for the integral

$$I = \int_C g^{\alpha\beta} R_{\alpha\beta} \sqrt{|g|}\, dx^0 \wedge \ldots \wedge dx^3$$

which generalizes the variational principle of general relativity. By subtracting from the initial connection the connection with zero torsion vector admitting the same parallelism, one obtains the equations of the field in a convenient form involving the $g_{\alpha\beta}$, the new connection $L^\gamma_{\alpha\beta}$ and the vector S_α. According to a study by Madame Tonnelat and Hlavaty, one of the partial systems[T32] provides the connection as an algebraic function of the $g_{\alpha\beta}$ and their first derivatives, except in an exceptional case which we will discard. We are thus led to define the field by the set ($g_{\alpha\beta}$, S_α) subject to the equations.

$$R_{\alpha\beta} - \frac{2}{3}(\delta_\alpha S_\beta - \delta_\beta S_\alpha) = 0 \qquad \delta_\varrho (g^{[\varrho\beta]} \sqrt{|g|}) = 0 \tag{9}$$

where $R_{\alpha\beta}$ is now relative to the new connection and considered as a function of the $g_{\lambda\mu}$ and their derivatives of the first two orders.

The existence of a variational principle leads, according to a classical procedure, to the existence of conservation identities.[T33] On the other hand, with the help of the change of coordinates already used in general relativity, one can see, without explicit calculations which would be intractable, which second-order derivatives relative to a hypersurface Σ occur in $R_{\alpha\beta}$. By a study too long to

to the surface then serve as initial data for "dynamical evolution" in that transverse direction, as determined by the field equations. This, by the by, is why he must have assumed the signature of the metric on V_5 to be $(+, -, -, -, -)$; otherwise, the boundary-value problem for induced initial-data on a hypersurface defined by orbits of the isometry would not be of Cauchy type.

T31. This last, about the exceptional surfaces in V_4, is somewhat cryptic, or at least severely elliptical. I believe he has in mind the following. The Cauchy problem on V_5 so constructed naturally induces a standard Cauchy problem in V_4 considered as a spacetime manifold (because the initial-data surface in V_5 is fixed by the orbits of the isometry, which themselves were used to define the equivalence relation whose modulus is V_4); the hyperbolic characteristics of the Cauchy problem in V_5 are the null cones of the pseudo-Riemannian metric (because we treat only the exterior, *i.e.*, "vacuum" or source-free, case); those null cones map by construction to the null cones on V_4, which are themselves the hyperbolic characteristics of the induced Cauchy problem there, since the induced Cauchy problem is that of the free Maxwell field.

[2]) ( ) and [ ] are the symbols for symmetrization and anti-symmetrization.

T32. I am not sure what 'partial systems' refers to.

T33. He may mean a number of things by this—most likely either Noether's Theorem or the mopping up of the diffeomorphism freedom (in a way expressed with peculiar clarity and insight by Schrödinger 1950, ch. xi).

be given here, these results allow us to establish that the system (9) is in involution and that by introducing an auxiliary condition of normalization of S_α, for example

$$\delta_\alpha \left(g^{(\alpha\beta)} S_\beta \sqrt{|g|}\right) = 0,$$

it has the same local mathematical coherence as the system of equations of general relativity; in particular the values of the $\delta_{00}(g^{(0\lambda)}\sqrt{|g|})$ on Σ ($x^0 = 0$) are unnecessary.

The main result of this study is that (9) admits the characteristic surfaces defined by the quadratic form of hyperbolic normal type with coefficients

$$l^{\alpha\beta} = g^{(\alpha\beta)}$$

and which differs from the quadratic form interpreted by EINSTEIN as defining the gravitational part. The bicharacteristics are here the null geodesics of

$$ds^2 = l_{\alpha\beta} dx^\alpha dx^\beta$$

where $l_{\alpha\beta}$ is dual to $l^{\alpha\beta}$. A second cone defined by a linear combination $\gamma_{\alpha\beta}$ of $l_{\alpha\beta}$ and $h_{\alpha\beta} = g_{(\alpha\beta)}$ also appears[1]). This study thus leads to the conclusion that *it is $l_{\alpha\beta}$ or a proportional tensor which must be interpreted as a gravitational tensor.*

2. Existence and Uniqueness for the Field Equations

5. The theorem of Mme Fourès[T34]

The preceding study leads naturally to the search, without any assumptions of analyticity, for theorems of existence and uniqueness at least local to the evolution problems of the different theories.[T35] This is a difficult problem in the theory of partial differential equations, and it is with a view to this problem that Madame FOURÈS has studied the following type of symmetry:

$$E_S \equiv A^{\alpha\beta} \delta_{\alpha\beta} W_S + f_S = 0 \quad (S = 1, 2, \ldots, \mathrm{N})$$

The W are unknown functions of 4 independent variables x^α, $A^{\alpha\beta}$ and f_S are given functions of W_R, $\delta_\alpha W_R$ and x^α, the quadratic form $A^{\alpha\beta} X_\alpha X_\beta$ is of normal hyperbolic type. On the hypersurface Σ ($x^0 = 0$) the CAUCHY data are:

$$W_S(x^i, 0) = \varphi_S(x^i) \qquad \delta_0 W_S(x^i, 0) = \psi_S(x^i)$$

The following hypotheses are laid down for the system (E_S) and the CAUCHY data:

1^0 In a neighborhood D_0 in Σ surrounding a point y with coordinates (y^i) and defined by $|x^i - y^i| \le d$, ϕ_S and ψ_S admit derivatives up to orders 6 and 5 respectively, continuous, bounded and satisfying the LIPSCHITZ conditions.[T36]

[1]) Added in proofs

T34. He refers to the result Fourès-Bruhat presented in the paper translated above.

T35. For a brief discussion of the importance of the lack of an assumption of analyticity, see my footnote T4 on p. 3 in the paper by Fourès-Bruhat translated above.

T36. The Lipschitz conditions for ϕ_S are (mildly abusing notation, but only for its own good): for each $0 \le S \le N$ and $0 \le n \le 6$, there exists a $K_{S,n} \ge 0$ such that $|\partial^n \phi_S(x) - \partial^n \phi_S(z)| \le K_{S,n}|x - z|$ for all $x, z \in D_0$, where '∂^n' denotes an n^{th}-order derivative; *mutatis mutandis* for ψ_S, with $0 \le n \le 5$.

2^0 In the domain D defined by $|x^i - y^i| \leq d$, $|x^0| \leq \varepsilon$ and for values of the unknowns such that:

$$|W_S - \varphi_S| \leq 1 \quad |\delta_i W_S - \delta_i \varphi_S| \leq 1 \quad |\delta_0 W - \psi_S| \leq 1,$$

a) the $A^{\alpha\beta}$ and f_S admit derivatives up to order 5, continuous, bounded and satisfying LIPSCHITZ conditions;

b) the quadratic form $A^{\alpha\beta} X_\alpha X_\beta$ is of normal hyperbolic type, the variable x^0 representing the temporal character and the variables x^i the spatial character ($A^{00} > 0$, $A^{ij} X_i X_j$ negative definite).

Under these conditions, Mme FOURÈS has established that *the* CAUCHY *problem for* (E_S) *admits one solution and only one in a certain neighborhood of* D_0. In the case where the $A^{\alpha\beta}$ does not contain the W nor their derivatives, which is the case in relativistic applications, a unity can be attained for all orders of differentiability.[T37]

It is impossible for me to sketch here the long study that leads to these results. I will limit myself to saying that a generalization of the classical KIRCHHOFF formulas[T38] play an essential role: in the linear case, these formulas express the values of the unknown functions at a point x_1 near D_0 based on their values on the surface of the characteristic conoid of vertex x_1 and from the CAUCHY data in the region of Σ inside this conoid.

6. Existence for the equations of EINSTEIN

The previous results apply in an elegant way to the EINSTEIN equations of general relativity by introducing *isothermal* coordinates.[T39] The idea is to associate to the EINSTEIN system an equation with only one unknown function f that admits the same characteristics as the system[1]). The simplest way to achieve this is to consider the equation of LAPLACE on V_4

$$\Delta f \equiv g^{\lambda\mu} (\delta_{\lambda\mu} f - \Gamma^{\varrho}_{\lambda\mu} \delta_\varrho f) = 0 .$$

A system (x^ρ) of local coordinates on V_4 is isothermal if

$$F^\varrho = \Delta x^\varrho = -g^{\lambda\mu} \Gamma^{\varrho}_{\lambda\mu} \qquad (10)$$

are zero for all ρ. It is easy to show, in particular with the help of the theorem of Mme FOURÈS, that given a local spacelike hypersurface Σ, it can always be considered as a coordinate manifold $x^0 = 0$ of a system of isothermal coordinates.

The quantities F^ρ appear in a simple way in the expression of the components of the RICCI tensor. Indeed, we have identically

$$R_{\alpha\beta} \equiv -G_{\alpha\beta} - L_{\alpha\beta} \qquad (11)$$

T37. I have no idea what he means by this. I suspect it is something straightforward and I am just being stoopid. Help.

T38. See https://encyclopediaofmath.org/wiki/Kirchhoff_formula.

T39. Lichnerowicz means harmonic coordinates, using the same terminology as Fourès-Bruhat in the paper translated above.

[1]) The theory and interpretation of isothermal coordinates are due to GEORGES DARMOIS. [trans. note: I presume this refers to the same work cited as '[2]' in the paper of Fourès-Bruhat translated above, since she also refers to it for the use of isothermal coordinates.]

with

$$G_{\alpha\beta} \equiv \frac{1}{2} g^{\lambda\mu} \delta_{\lambda\mu} g_{\alpha\beta} + H_{\alpha\beta} \tag{12}$$

and

$$L_{\alpha\beta} \equiv \frac{1}{2} (g_{\alpha\mu} \delta_\beta F^\mu + g_{\beta\mu} \delta_\alpha F^\mu) \tag{13}$$

the $H_{\alpha\beta}$ designating polynomials with respect to $g_{\lambda\mu}$, $g^{\lambda\mu}$ and their first derivatives. It is the structure of $R_{\alpha\beta}$ thus revealed that we will exploit. To simplify the expressions, we will limit ourselves to equations without a cosmological constant.

Let us consider a hypersurface Σ in V_4 carrying the CAUCHY data. On Σ $(x^0 = 0)$ these data satisfy

$$(S^0_\alpha)_{x^0=0} = 0\,. \tag{14}$$

Moreover, since we propose to use coordinates isothermal relative to the sought-after metric, we will assume, without loss of generality, that they satisfy

$$(F^\mu)_{x^0=0} = 0\,. \tag{15}$$

We propose to study existence and uniqueness of the problem of CAUCHY for the system of EINSTEIN

$$R_{\alpha\beta} \equiv -G_{\alpha\beta} - L_{\alpha\beta} = 0$$

whose first members are related by conservation identities.[T40] The stages of the reasoning are the following.

1^0 *Resolution of the* CAUCHY *problem for the system* $G_{\alpha\beta} = 0$.[T41] This system is of Mme FOURÈS's type; we will make the following hypotheses in a neighborhood D_0 of Σ.

a) The CAUCHY data $g_{\alpha\beta}$ and $\delta_0 g_{\alpha\beta}$ admit partial derivatives up to orders 5 and 4 respectively, which are continuous, bounded and satisfy LIPSCHITZ conditions.

b) On Σ, the form $g^{\alpha\beta} X_\alpha X_\beta$ is of normal hyperbolic type with $g^{00} > 0$ and $g^{ij} X_i X_j$ negative definite. Under these conditions, the CAUCHY problem for $G_{\alpha\beta} = 0$ has a unique solution in a neighborhood of D_0, a solution which admits partial derivatives up to order 4, continuous and bounded.

2^0 *The solution found satisfies the isothermal conditions.* In virtue of (14) and (15) there results

$$(\delta_0 F^\mu)_{x^0=0} = 0\,.$$

On the other hand, for any solution of $G_{\alpha\beta} = 0$, the conservation identities reduce to the equations

T40. I take it he means the Bianchi identities.

T41. The most salient fact about the condition $G_{\alpha\beta} = 0$ is that it results in an expression for the Ricci tensor (and so the Ricci scalar and so the Einstein tensor) with derivatives of the metric only up to first order. One thus has a great deal of freedom in setting terms to zero by a judicious choice of coordinate system.

$$g^{\alpha\beta}\,\delta_{\alpha\beta}F^{\mu} + P^{\mu}(\delta_{\alpha}F^{\varrho}) = 0$$

where P^{μ} is linear with respect to $\delta_\alpha F^\rho$, the coefficients being polynomials in $g_{\alpha\beta}$, $g^{\alpha\beta}$ and their first derivatives. This system is of Mme FOURÈS's type and the uniqueness of the corresponding CAUCHY problem implies $F^\mu = 0$.

Thus the solution found for $G_{\alpha\beta} = 0$ is a solution of the EINSTEIN system $R_{\alpha\beta} = 0$ compatible with the CAUCHY data and related to isothermal coordinates. We have thus obtained a local existence theorem for the EINSTEIN system, without any assumption of analyticity.

7. Uniqueness for the equations of EINSTEIN

It is clear that the uniqueness of the problem of CAUCHY for the system of EINSTEIN must be understood in a completely different sense than the usual uniqueness, which is used here, for example for the system $G_{\alpha\beta} = 0$. We understand, for the EINSTEIN system, uniqueness [to mean] *up to a coordinate change preserving the numerical values of the coordinates of any point of* Σ *as well as the* CAUCHY *data on* Σ.[T42] In this sense, it is possible to speak of "physical uniqueness".

To establish this physical uniqueness, it must be shown that any solution for the CAUCHY problem with $R_{\alpha\beta} = 0$ can be deduced, by means of a change of coordinates satisfying the previous hypotheses, from the unique solution of the same problem for $G_{\alpha\beta}$. This uniqueness was previously established by STELLMACHER following the work of FRIEDRICHS and HANS LEWY.

I have developed here methods and results for the equations of general relativity. This method can be adapted, without major difficulties, to the theory of JORDAN-THIRY. By contrast, the analogous theorems for the non-symmetric unitary theory present difficulties related to the properties of the "isothermal coordinates" in that theory.[T43]

3. Models of the Universe and Global Problems

8. Models of the universe in general relativity

The previous studies were purely local, but in fact the fundamental mathematical problems of any relativistic field theory must be essentially global in nature.[T44]

I will first limit myself to gravitation and the theory of general relativity [1]). The question that arises is the following: *when is a problem of gravitation actually solved?*[T45]

T42. This is far more restrictive than is actually needed. Is he worried about problems related to the Hole Argument?

T43. I am curious whether the non-symmetric unitary theory admits "analogous *theorems*", or whether he really means to say that the analogous propositions, which may or may not be theorems in the theory, cannot be proved in the same way, *viz.*, using isothermal coordinates. Any experts in the audience?

T44. I think we must be cautious in trying to understand what he means by 'global', and in particular not assume that he means what mathematical and theoretical physicists in the 1960s and later meant by it.

[1]) without a cosmological constant for simplicity.

T45. I have a feeling we're about to get into some heavy shit. This feeling is strengthened by the lack of mathematical formulæ, nay, even mathematical symbols, in the immediately subsequent text.

I propose to call *model of the universe* a manifold V_4 with an everywhere regular metric, satisfying the equations of EINSTEIN for the different cases and possibly [under the imposition of] asymptotic conditions.[T46] In the neighborhood of the timelike hypersurfaces separating the regions occupied by energy[T47] from the vacuum regions, there must exist, in accordance with our general axioms, admissible local coordinates such that, when crossing the hypersurfaces, the corresponding potentials and their first derivatives are continuous, the second derivatives being discontinuous.[T48]

It is when it is possible to construct such a model of the universe that the exterior field can be considered as *effectively produced* by the different masses or energy distributions in motion, and it is the connection of the internal fields of the different distributions with a single field that ensures *the interdependence of the motions.*[T49] What is called the principle of geodesics is an easy corollary of this fact and the fundamental tool is basically *the continuity, when crossing* Σ *(*$x^0 = 0$*), of the quantities* S^0_α.[T50]

Only such a model of the universe is susceptible to physical interpretation.[T51] In a region Δ_0

T46. I think 'different cases' must mean "both vacuum and non-vacuum", *i.e.*, vanishing and non-vanishing stress-energy tensor, in order to make sense of what comes next.

T47. Literally "swept by energy" ('balayées par l'énergie'), which either is Lichnerowicz waxing poetic or else is a technical usage I am unfamiliar with.

T48. He seems to be envisaging a universe that consists of regions of vacuum surrounding regions filled with matter—empty space filled with dust, stars, and so on, and no CMBR, since that was yet unknown. In such a universe, the boundaries of the regions swept by energy, *viz.*, filled with matter ("world tubes"), would be timelike hypersurfaces.

T49. We have entered the heavy shit. He seems to be gesturing at several complex ideas at once, the clearest of which to me are: that something like Mach's Principle can be said to hold only in such spacetimes; that the metric field plays the role of coordinator of inertial motion for all material bodies; and that the metric field mediates all interactions among material bodies. Again, the unspoken target seems to be Einstein's views on matter and motion. (See my footnote T51 on p. 16 below.) Lichnerowicz seem to think that, on Einstein's view, strictly speaking, matter can never serve as a source for the metric field, as matter is always singular. For coeval discussion of Mach's Principle directly relevant to all the issues Lichnerowicz raises here, see Pirani (1956), the lecture delivered by him at the same conference, in the session immediately following that in which Lichnerowicz delivered this. Indeed, the discussion by Pirani illuminates many issues that this essay by Lichnerowicz importantly but only implicitly bears on, such as the struggle of researchers in the 1950s to develop methods to handle the representation of physical phenomena in general relativity with global features and properties.

T50. Recall that

$$S^0_\alpha = \chi T^0_\alpha$$

so he is in effect claiming, rightly, that the geodesic principle follows from conservation of stress-energy, but he also seems to me to be claiming, at best confusedly or misleadingly and at worst illegitimately, that the principle is essentially tied to the peculiar character of universe-model spacetimes.

T51. Damn! Shit just got real. Where is this coming from? What are Lichnerowicz's philosophical commitments? I guess we first have to figure out what "such a model" means—just regularity, as suggested by the next sentence? But then why go into detail about the interior and exterior regions? A fully vacuum spacetime can be regular according to the definition Lichnerowicz gave in §1 of the paper (p. 6 above) but, I suspect, would for Lichnerowicz not be susceptible to physical interpretation. Now that I've thought about the matter more, however, it occurs to me that Lichnerowicz's true target is Einstein's views on matter and motion in general relativity, particularly as worked out by Einstein and Grommer (1927) and Einstein et al. (January 1938), and explained lucidly by Dennis in Lehmkuhl (2019), according to which matter is modeled as a singularity. That would explain the emphasis on regularity, and also makes sense of the otherwise baffling proposal in the second subsequent sentence, to replace a non-regular metric in a region of spacetime with a regular metric arising from a matter distribution imposed on the region. And this is borne out by the discussion of §10 below (p. 19). Even putting all that aside, it also seems to be the case that Lichnerowicz is not a fan of severe idealization in the modeling of the universe—not, perhaps, a standard attitude for one manifesting the spirit of the mathematical physicist.

of V_4 where there is no regularity, a metric is not susceptible of any interpretation. One should, in order to try to arrive at a model of the universe, see whether it is possible to *furnish* such a region,[T52] *i.e.*, to choose a hypersurface Σ bounding a region Δ containing Δ_0 and to construct in Δ an energy distribution and a metric related by the equations of EINSTEIN, the metric being everywhere regular in Δ and connecting along [the boundary of] Δ with the previously given metric. It should be noted that such a problem is essentially global in nature and has some analogy with classical problems in hydrodynamics. About the solution of such problems, we know almost nothing.

In a model of a universe, in the sense that we have defined it, it should be impossible to introduce new energy distributions whose associated metrics are connected with the external field. We must therefore study the validity, in relativity, of the following proposition: *The introduction of energy distributions in a given external field can only be done in domains where this field is not regular* (proposition A).[T53]

Closely related to this proposition is the following: *A model of the universe constituted by an everywhere regular external field must be trivial, that is, locally Euclidean* (proposition B).[T54]

The introduction of an electromagnetic field in general relativity or the JORDAN-THIRY theory leads to analogous concepts and statements concerning the set of two fields.

Such propositions do not seem valid under the general axioms I have indicated, as shown by a few somewhat teratological[T55] counterexamples. But, as we shall see, they are valid for stationary fields and consequently for fields sufficiently close to stationary fields, which appears to be reassuring.

A definition of what would be called a universe model in a non-symmetric unitary theory has never been given. If one wants to avoid the artificial introduction of sources – and this was obviously EINSTEIN's intention – it would be advisable to pass, if I may say so, to the second member and to physically interpret certain terms of the field equations, the new first members still satisfying conservation conditions.[T56] Nothing worthwhile has yet been done in this direction.[T57]

T52. I like this metaphor, of supplying regularizing furniture to efface the singularity of matter of Einstein's view; one may perhaps be permitted to speak here of an urbane Cosmic Interior Decorator, a much happier profession than Penrose's finical Cosmic Censor.

T53. We need to talk about this.

T54. Ditto.

T55. Ouch! We also need to talk about this. What are these monstrous spacetimes he has in mind? Proposition B in particular, if I understand what he is saying, is just trivially false, and one needs nothing monstrous to show it—unless he considers interior Schwarzschild, cylindrical gravitational waves and the menagerie of Weyl and Levi-Civita solutions to be monstrous. If so, perhaps fair enough. In any event, it seems again as if Lichnerowicz is hostile to what may reasonably be thought of as extreme idealizations in the modeling of relativistic systems.

T56. The awkwardness of the translation comes from the fact that I feel like I grasp only dimly at best what Lichnerowicz is saying here—something like "move some stuff from the lefthand side of the field equations to the righthand side, and re-interpret the new stuff as matter, all in such a way that both sides are still covariantly conserved"—but I can't be sure, in large part because I have no understanding of the non-symmetric unitary theory at all. Help?

T57. I feel you, brother. (See my previous footnote.)

9. Global problems for stationary fields

In general relativity, a field is stationary if the Riemannian manifold V_4 admits a 1-parameter group of isometries with timelike orbits (time lines).[T58] The metric can be written:

$$ds^2 = \xi^2 \left[(dx^0)^2 + 2\, \varphi_i\, dx^0\, dx^i\right] + g_{ij}(x)\, dx^i\, dx^j$$

where the potentials are independent of the time variable x^0 ($\xi^2 = g^{00} > 0$). These assumptions correspond physically to a permanent steady-state.

I further assume, although it is not strictly necessary, that V_4 is homeomorphic to the topological product of a 3-dimensional manifold and a line, where the factor-manifold W_3 of V_4 can be represented by $x^0 =$ const., the factor-lines being the time-lines. The W_3 are equipped with a negative definite metric with coefficients:

$$g^*_{ij} = g_{ij} - \frac{g_{0i}\, g_{0j}}{g_{00}}\,.$$

By local calculations one shows on W_3

$$\xi\, R^0_0 = \mathrm{div}^* h \qquad (16)$$

$$\Delta^* \xi = \frac{\xi^3}{2} H^2 \quad \text{(pour un champ extérieur)} \qquad (17)$$

$$\xi\, \varphi_i\, R^i_0 = \frac{\xi^3}{2} H^2 - \mathrm{div}^*\, p \qquad (18)$$

where h and p are vectors in W_3 depending only on the potentials and their first derivatives, and where $H^2 = 0$ expresses the fact that the congruence of the time lines is a normal congruence for spatial sections.[T59]

With the help of (16) one can easily establish proposition A for stationary fields.[T60] As for proposition B, one assumes W_3 to be compact or to admit a domain at infinity with asymptotically Euclidean behavior; its proof then uses the relations (17) and (18) and proceeds by reduction from the case of the stationary field to the case of the static field, in the sense of Levi-Civita, *i.e.*, $H^2 = 0$.[T61] The results thus obtained can be extended without difficulty to the case where there is an electromagnetic field or to the theory of Jordan-Thiry.

In the absence of a cosmological constant, there cannot exist a model of a stationary universe with compact W_3.[T62] For a stationary universe model with an infinite domain for which the flow

T58. This is more restrictive than the modern definition, which demands only that the orbits of the isometries (Killing fields) be asymptotically timelike, in order to accommodate spacetimes such as Schwarzschild, where the stationary Killing field is null on the event horizon and spacelike inside. Lichnerowicz seems to lose nothing by the restriction, since he would not have approved of such terata.

T59. $H^2 = 0$ iff the subspaces in the tangent planes orthogonal to the vectors tangent to the time lines are integrable in the sense of Frobenius. This follows from (17).

T60. We need to talk about this.

T61. Ditto.

T62. True. And a deep general result, not tied to particular solutions or types of matter or types of configuration or symmetry. That's the sort of thing, folklore has it, that didn't get started until the Golden Age of Global Structure in the 1960s, at the hands of Penrose, Geroch, Misner, Hawking, *et al.* Note also that it is independent of the assumptiom of an energy condition, perhaps the most typical feature of the global theorems of the Golden Age—happenstance or indicative of something deeper? I would love to track down when such general theorems started appearing, from whom, and why.

lines[T63] inside the masses coincide with the time lines, we can deduce by integration of (18) that $H^2 = 0$ everywhere.[T64] It follows in particular that the postulates usually introduced for the construction of the SCHWARZSCHILD universe model are overabundant.[T65]

10. Approximations and equations of motion

Even if many of the rigorous problems of relativistic theory seem to be beyond our powers, it is possible to treat by approximations the problem of the motion of n gravitating masses.

The coordinates are assumed to be isothermal, the metric quasi-Euclidean and the behavior asymptotically Euclidean, and the potentials are developed in powers of c^{-2}. For the initial technique of EINSTEIN, INFELD, HOFFMANN which used a representation of the masses as pure singularities of the external field, a representation which could be spurious, it is preferable to substitute a technique where the momentum-energy tensor plays its role.[T66] Such a technique, which gives satisfactory results, was initiated by FOCK and by PAPAPETROU and has been developed more rigorously by Madame HENNEQUIN. The equations of motion of the masses arise essentially from the integration, in the tubes swept out by them, of divergences suggested by the first members of the conservation conditions, so as to express the fact that the quantities S^0_α are zero at the boundary of those tubes.

I will not go into the details of this technique, but I will point out that the same procedure has been applied to the equations of the JORDAN-THIRY theory and that the approximations obtained suggest the following interpretation, which differs from the one initially given by the authors of the theory: in the notation of §3 [p. 10 above], it is $\overline{ds}^2 = \xi\, ds^2$ that represents the gravitational metric;[T67] the electromagnetic field is represented by the set of two proportional tensors

$$\overline{F}_{\alpha\beta} = F_{\alpha\beta} \qquad \overline{H}_{\alpha\beta} = \xi^3 F_{\alpha\beta}$$

where $\bar{F}$ has zero external derivative and ξ^3 plays the role of a dielectric permittivity of the vacuum. In the exterior unitary case the field equations are written with the $\overline{ds}^2$ metric:

$$\bar{S}_{\alpha\beta} + K_{\alpha\beta} = \frac{\beta^2}{2}\bar{\tau}_{\alpha\beta}$$

$$\bar{\nabla}_\beta(\bar{H}^\beta_\alpha) = 0$$

$$\bar{\Delta}\log\xi + \frac{\beta^2}{2}(\overline{F}, \overline{H}) = 0$$

where $K_{\alpha\beta}$ depends only on the first derivatives of $\log\xi$ and where $\bar{\tau}_{\alpha\beta}$ is the momentum-energy tensor of the electromagnetic field

T63. 'Lignes de courant' clearly means here something like "worldlines of mass particles".

T64. Again true and deep, in so far as it plays global topological properties off of local metric ones. Note as well again that, because the argument depends only on the fact that the values of the mixed timelike-spacelike components of the Ricci tensor are not all zero, it cannot depend on an energy condition.

T65. No idea what "the postulates usually introduced" for Schwarzschild are, but it seems clear that Lichnerowicz just showed some of them to be otiose—those of you who know history, please help!

T66. YES! I knew it!!! He is obsessed with Einstein's views on matter and motion. He rather wants to take general relativity seriously on its own terms.

T67. It is not clear to me what Lichnerowicz means by "the gravitational metric". The one governing the inertial paths of massive bodies and light rays? The one that measures proper time? What?

$$\tilde{\tau}_{\alpha\beta} = \frac{1}{4} g_{\alpha\beta} \bar{F}_{\lambda\mu} \bar{H}^{\lambda\mu} - \bar{F}_{\alpha\varrho} \bar{H}^{\varrho}_{\beta} .$$

The gravitational factor $\beta^2/2$ is then constant.

We have tried to review the mathematical issues presented by the relativistic field equations. Much work remains to be done.

Diskussion – Discussion

D. van Dantzig: 1. Since the gravitational equations are nonlinear, the cone of the bicharacteristics will depend in general on the considered solution. Is it known under which conditions one can be sure that, when extending a local solution, the signature of g_{ij} will be preserved, especially that the bicharacteristic cone will not become degenerate?

2. Can the solution to the Cauchy data be represented by means of ordinary integrals, either on the cone or inside the cone (or a combination of both), or are difficulties of the type of Hadamard, where one must take the "finite part" of an infinite integral, inevitable?[T68]

Mme. Y. Fourès-Bruhat: 1. It is not known, in the general case, under which conditions one can extend a given solution. This would solve the problem of the existence of regular global solutions, a problem whose answer would be very important to know, but is certainly very difficult.

2. The solution is obtained by solving integral equations (based on ordinary integrals defined on the bicharacteristic cone) by successive approximations. The solution depends on the initial data inside the cone (wave propagation, in general spread out).[T69]

Mme. A. Tonnelat: I would like to point out that it is also possible to define isothermal coordinate systems in the non-symmetric theory ($g^{\mu\nu}\Gamma^{\rho}_{\mu\nu} = 0$). Their use should lead to a great number of simplifications. Nevertheless, to my knowledge, no serious application of this choice of coordinates has been proposed.

Bibliography

The various memoirs of Einstein and his collaborators and Einstein and Pauli Ann. of Math. *44*, p. 131-138 (1943).

Bergmann, P. G., *Introduction to the theory of relativity.*, Prentice Hall, New-York (1950).

Darmois, G., *Les équations de la gravitation einsteinienne*, Mém. Sc. Math, Paris (1927).

Fourès, Mme Y., Acta Math. 88, pp. 141–225 (1952).

Hadamard, J., *Leçons sur la propagation des Ondes* (Hermann, Paris 1903).

Hennequin, Mme, *Différentes notes aux Comptes rendus de Académie des Sciences de Paris* (1954 and 1955).

Hilbert, D., Math. Ann. *92*, pp. 1–32 (1924).

Jordan, P., Ann. Physik *32*, p. 219 (1947).

Lichnerowicz, A., *Problèmes globaux en mécanique relativiste* (Hermann, Paris, 1939); C. R. Acad. Sc. *222*, p. 432 (1946). *Les théories relativistes de la gravitation et de electromagnetisme* (Masson, Paris 1955).

T68. Anyone more historically knowledgeable than I know what problems this dude is talking about, and what Hadamard's solution consisted of?

T69. I'm not sure what that last qualification, 'diffusées', means in this context.

Racine, Ch., *Le problème des n corps en théorie de la relativité* (These, Paris 1934).

Stellmacher, K., Math. Ann. *115*, p. 136–152 (1938).

Thiry, Y., C. R. Acad. Sc. *226*, p. 216 and 1881 (1948), Journ. math. p. et appl. *30*, p. 275–396 (1951).

Translation References

Callender, Craig. 2017. *What Makes Time Special?* Oxford: Oxford University Press.

Einstein, A., and J. Grommer. 1927. "Allgemeine Relativitätstheorie und Bewegungsgesetz". *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse,* number I, 2–13.

Einstein, Albert, Léon Infeld, and Banesh Hoffmann. January 1938. "The Gravitational Equations and the Problem of Motion". *Annals of Mathematics,* Second, 39, number 1 (): 65–100. doi:10.2307/1968714.

Fourès-Bruhat, Y. 1956. "Le probléme de Cauchy dans la théorie relativiste de l'électromagnétisme et dans la théorie unitaire de Jordan-Thiry". In Mercier and Kervaire 1956, 76–78.

Geroch, Robert. 1969. "Spinor Structure of Space-Times in General Relativity. I". *Journal of Mathematical Physics* 9 (11): 1739–1744. doi:10.1063/1.1664507.

Geroch, Robert, and Gary T. Horowitz. 1979. "Global Structure of Spacetimes". Chapter 5 in *General Relativity: An Einstein Centenary Survey,* edited by Stephen W. Hawking and Werner Israel, 212–293. Cambridge: Cambridge University Press.

Lehmkuhl, D. 2019. "General Relativity as a Hybrid Theory: The Genesis of Einstein's Work on the Problem of Motion". *Studies in History and Philosophy of Modern Physics* 67:176–190. arXiv:1803.09872 [gr-qc], doi:10.1016/j.shpsb.2017.09.006.

Lichnerowicz, A. 1956. "Problèmes généraux d'intégration des équations de la relativité". In Mercier and Kervaire 1956, 176–191.

Mercier, André, and Michel Kervaire, editors. 1956. *Fünfzig Jahre Relativitätstheorie.* Helvetica Physica Acta Supplementum, IV. Basel: Séminaire de Physique Théorique de l'Université de Berne, Birkhäuser.

Pirani, F. A. E. 1956. "On the Definition of Inertial Systems in General Relativity". In Mercier and Kervaire 1956, 198–203.

Schouten, J. 1954. *Ricci-Calculus: An Introduction to Tensor Analysis and Its Geometrical Applications.* Second. Die Grundlehren Der Mathematischen Wissenschaften in Einzeldarstellung, X. First edition published in 1923, in German. The second edition was written in English, and contains significant additions to the first edition. Berlin: Springer-Verlag.

Schrödinger, E. 1950. *Space-Time Structure.* Cambridge: Cambridge University Press.

Steenrod, N. 1951. *The Topology of Fibre Bundles.* Princeton Mathematical Series 14. Princeton, NJ: Princeton University Press.