A Simple Proof of the Uniqueness of the Einstein Field Equation in All Dimensions†

Erik Curiel‡

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ABSTRACT

The standard argument for the uniqueness of the Einstein field equation is based on Lovelock's Theorem, the relevant statement of which is restricted to four dimensions. I prove a theorem similar to Lovelock's, with a physically modified assumption: that the geometric object representing curvature in the Einstein field equation ought to have the physical dimension of stress-energy. The theorem is stronger than Lovelock's in two ways: it holds in all dimensions, and so supports a generalized argument for uniqueness; and it does not assume that the desired tensor depends on the metric only up second-order partial-derivatives, that condition being a consequence of the proof. This has consequences for understanding the nature of the cosmological constant and Lanczos-Lovelock theories of higher-dimensional gravity. Another consequence of the theorem is that it makes precise the sense in which there can be no gravitational stress-energy tensor in general relativity. Along the way, I prove a result of some interest about the second jet-bundle of the bundle of metrics over a manifold.

The Einstein field equation, $G_{ab} = 8\pi\gamma T_{ab}$ (where γ is Newton's gravitational constant) consists of an object representing the curvature of spacetime (the Einstein tensor, G_{ab}) equated with the

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‡**Author's address**: Munich Center for Mathematical Philosophy, Ludwig-Maximilians-Universität; Black Hole Initiative, Harvard University; Smithsonian Astrophysical Observatory, Radio and Geoastronomy Division; **email**: erik@strangebeautiful.com

stress-energy tensor of material fields (T_{ab}). The standard proof of the uniqueness of the equation invokes the classic theorem by Lovelock (1972),

Theorem 1 *Let* $(\mathcal{M},\, g_{ab})$ *be a four-dimensional spacetime. In a coordinate neighborhood of a point* $p \in \mathcal{M}$*, let* $\Theta_{\alpha\beta}$ *be the components of a tensor concomitant of* $\{g_{\lambda\mu};\ g_{\lambda\mu,\nu};\ g_{\lambda\mu,\nu\rho}\}$ *such that*

$$\nabla^n \Theta_{nb} = 0.$$

where ∇_a *is the derivative operator associated with the metric* g_{ab}*. Then*

$$\Theta_{ab} = qG_{ab} + rg_{ab},$$

where G_{ab} *is the Einstein tensor, and* q *and* r *are constants.*

The restriction to four dimensions is essential for the result. In higher dimensions, there are other tensors satisfying the theorem. (Those tensors are not linear in the second-order partial-derivatives of the metric as the Einstein tensor is.) Those tensors form the basis of so-called Lovelock gravity theories (Lovelock 1971; Padmanabhan and Kothawala 2013).

In this note, I sketch the proof of the following:

Theorem 2 *The only two covariant-index, divergence-free concomitants of the metric that are homogeneous of weight zero are constant multiples of the Einstein tensor.*

There is a subtle but important difference between Lovelock's original theorem and my result, one with interesting consequences. Lovelock did not require the concomitant to be homogeneous of weight zero, the assumption capturing the idea that the desired concomitant has the physical dimension of stress-energy (as I explain below). The theorem is thus weaker than Lovelock's in one sense. It also, however, makes it stronger in two important senses: the assumption of being second-order in the metric is not required, but follows from the proof; and perhaps more importantly, my result does not depend on the dimension of the manifold, proving uniqueness of the Einstein field equation in all dimensions, not just four.[1]

First, I lay down the needed definitions. (From hereon, I use the Geroch-Newman-Penrose abstract-index notation; see Wald 1984.)

1. Navarro and Sancho (2008), in a paper I was unware of when I did this work, prove a similar theorem based on similar methods. (I thank Navarro for bringing it to my attention.) The most salient differences between that work and mine are: they do not provide detailed arguments for the relation between the scaling of the metric and the idea and fixing of the physical dimension of quantities; their arguments, being based on category-theoretic notions of sheaves of sections of bundles and natural (in the sense of category theory) constructions on those sheaves, do not have the straightforward and intuitively perspicuous physical interpretation of mine, especially in the context of general relativity; as a consequence, all of my constructions and arguments are considerably simpler and more straightforwardly geometrical in character; although theorem 6 is implicit in their work, it is difficult to tease out, because of the algebraic complexity of their machinery, and in particular it is difficult to see the central role played by my lemma 9. Finally, and most importantly, they do not discuss, as I do, the immediate and important bearing the central result has on the possibility of defining a gravitational stress-energy tensor, and on the appropriate physical interpretations of the cosmological constant and Lanczos-Lovelock theories of gravity.

Definition 3 *For two fiber bundles* $(\mathcal{B}_1,\ \mathcal{M},\ \pi_1)$ *and* $(\mathcal{B}_2,\ \mathcal{M},\ \pi_2)$ *over the same base space* $\mathcal{M}$*, a mapping* $\chi : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ *is a* concomitant *if*

$$\chi(\phi_1^*(u_1)) = \phi_2^*(\chi(u_1))$$

for all $u_1 \in \mathcal{B}_1$ *and all diffeomorphisms* ϕ *from* $\mathcal{M}$ *to itself, where* ϕ_i^* *is the natural diffeomorphism induced by* ϕ *on the bundle space* $\mathcal{B}_i$*.*

This definition can be generalized to take account of how a concomitant can depend on differentials of the fiber bundle $\mathcal{B}$ that is its domain, based on the n^{th}-order jet bundle of $\mathcal{B}$, $J^n\mathcal{B}$. There is a natural projection $\theta^{n,m} : J^n\mathcal{B} \rightarrow J^m\mathcal{B}$ (for $0 < m < n$), characterized by taking the Taylor expansion that defines the n-jet and "dropping all terms above order m".

Definition 4 *An* n^{th}-order concomitant *(n a strictly positive integer) from* $\mathcal{B}_1$ *to* $\mathcal{B}_2$ *(bundles over the same base space* $\mathcal{M}$*) is a smooth mapping* $\chi : J^n\mathcal{B}_1 \rightarrow \mathcal{B}_2$ *such that for all* $u \in J^n\mathcal{B}_1$ *and diffeomorphisms* ϕ *from* $\mathcal{M}$ *to itself*

1. $\phi_2^*(\chi(u)) = \chi(\phi_n^*(u))$
2. *there is no* $(n-1)^{th}$*-order concomitant* $\chi' : J^{n-1}\mathcal{B}_1 \rightarrow \mathcal{B}_2$ *satisfying* $\chi(u) = \chi'(\theta^{n,n-1}(u))$ *for all* $u \in J^n\mathcal{B}_1$

A zeroth-order concomitant (or just 'concomitant' for short, when no confusion will arise) is one satisfying definition 3. An important property of concomitants is that, in a limited sense, they are transitive.

Proposition 5 *If* $\chi_1 : J^n\mathcal{B}_1 \rightarrow \mathcal{B}_2$ *is an* n^{th}*-order concomitant and* $\chi_2 : \mathcal{B}_2 \rightarrow \mathcal{B}_3$ *is a smooth mapping, where* $\mathcal{B}_1$*,* $\mathcal{B}_2$ *and* $\mathcal{B}_3$ *are bundles over the same base space, then* $\chi_2 \circ \chi_1$ *is an* n^{th}*-order concomitant if and only if* χ_2 *is a zeroth-order concomitant.*

This follows immediately from the definition of n^{th}-order concomitants and the properties of the natural lifts of diffeomorphisms from a base space to a jet bundle. Finally, a concomitant is *homogeneous of weight* w if for any constant scalar field ξ

$$\chi(\phi_1^*(\xi u)) = \xi^w \phi_2^*(\chi(u))$$

This definition makes sense, as we consider only bundles of linear and affine objects in this paper.

We now explicate the structure of the first two jet bundles of the bundle of metrics over a manifold. Two metrics g_{ab} and h_{ab} are in the same 1-jet at a point if and only if they have the same associated covariant derivative operator at that point. To see this, first note that, if they are in the same 1-jet, then $\hat{\nabla}_a(g_{bc} - h_{bc}) = 0$ at that point for all derivative operators. Thus, for the derivative operator ∇_a associated with, say, g_{ab}, $\nabla_a(g_{bc} - h_{bc}) = 0$, but $\nabla_a g_{bc} = 0$, so $\nabla_a h_{bc} = 0$ at that point as well. Similarly, if the two metrics are equal and share the same associated derivative operator ∇_a at a point, then $\hat{\nabla}_a(g_{bc} - h_{bc}) = 0$ at that point for all derivative operators, since their difference will be identically annihilated by ∇_a, and $g_{ab} = h_{ab}$ at the point by assumption. Thus they are in the same 1-jet. This proves that all and only geometrically relevant information contained in the 1-jets of Lorentz metrics on $\mathcal{M}$ is encoded in the fiber bundle over spacetime the values of the fibers of which are ordered pairs consisting of a metric and the metric's associated derivative operator at a spacetime point.

The second jet bundle over $\mathcal{B}_{\mathrm{g}}$ has a similarly interesting structure. Clearly, if two metrics are in the same 2-jet, then they have the same Riemann tensor at the point associated with the 2-jet, since the result of doubly applying to it an arbitrary derivative operator (not the Levi-Civita one associated with the metric) at the point yields the same tensor. Assume now that two metrics are in the same 1-jet and have the same Riemann tensor at the associated spacetime point. If it follows that they are in the same 2-jet, then essentially all and only geometrically relevant information contained in the 2-jets of Lorentz metrics on $\mathcal{M}$ is encoded in the fiber bundle over spacetime the points of the fibers of which are ordered triplets consisting of a metric, the metric's associated derivative operator and the metric's Riemann tensor at a spacetime point. To demonstrate this, it suffices to show that if two Levi-Civita connections agree on their respective Riemann tensors at a point, then the two associated derivative operators are in the same 1-jet of the bundle whose base-space is $\mathcal{M}$ and whose fibers consist of the affine spaces of derivative operators at the points of $\mathcal{M}$ (because they will then agree on the result of application of themselves to their difference tensor, and thus will be in the 2-jet of the same metric at that point).

Assume that, at a point p of spacetime, $g_{ab} = \tilde{g}_{ab}$, $\nabla_a = \tilde{\nabla}_a$ (the respective derivative operators), and $R^a{}_{bcd} = \tilde{R}^a{}_{bcd}$ (the respective Riemann tensors). Let $C^a{}_{bc}$ be the symmetric difference-tensor between ∇_a and $\tilde{\nabla}_a$, which is itself 0 at p by assumption. Then by definition $\nabla_{[b}\nabla_{c]}\xi^a = R^a{}_{bcn}\xi^n$ for any vector ξ^a, and so at p

$$\begin{aligned} R^c{}_{abn}\xi^n &= \nabla_{[a}\tilde{\nabla}_{b]}\xi^c \\ &= \nabla_a(\nabla_b\xi^c + C^c{}_{bn}\xi^n) - \tilde{\nabla}_b\nabla_a\xi^c \\ &= \nabla_a\nabla_b\xi^c + \nabla_a(C^c{}_{bn}\xi^n) - \nabla_b\nabla_a\xi^c - C^c{}_{bn}\nabla_a\xi^n + C^n{}_{ba}\nabla_n\xi^c \end{aligned}$$

but $\nabla_b\nabla_c\xi^a - \nabla_c\nabla_b\xi^a = 2R^a{}_{bcn}\xi^n$ and $C^a{}_{bc} = 0$, so expanding the only remaining term gives

$$\xi^n\nabla_a C^c{}_{bn} = 0$$

for arbitrary ξ^a and thus $\nabla_a C^b{}_{cd} = 0$ at p; by the analogous computation, $\tilde{\nabla}_a C^b{}_{cd} = 0$ as well. It follows immediately that ∇_a and $\tilde{\nabla}_a$ are in the same 1-jet over p of the affine bundle of derivative operators over $\mathcal{M}$. We have proven

Theorem 6 *$J^1\mathcal{B}_g$ is naturally diffeomorphic to the fiber bundle over $\mathcal{M}$ whose fibers consist of pairs (g_{ab}, ∇_a), where g_{ab} is the value of a Lorentz metric field at a point of $\mathcal{M}$, and ∇_a is the value of the covariant derivative operator associated with g_{ab} at that point. $J^2\mathcal{B}_g$ is naturally diffeomorphic to the fiber bundle over $\mathcal{M}$ whose fibers consist of triplets $(g_{ab}, \nabla_a, R_{abc}{}^d)$, where g_{ab} is the value of a Lorentz metric field at a point of $\mathcal{M}$, and ∇_a and $R_{abc}{}^d$ are respectively the covariant derivative operator and the Riemann tensor associated with g_{ab} at that point.*

It follows immediately that there is a first-order concomitant from $\mathcal{B}_{\mathrm{g}}$ to the geometric bundle $(\mathcal{B}_{\nabla}, \mathcal{M}, \pi_{\nabla}, \iota_{\nabla})$ of derivative operators, *viz.*, the mapping that takes each Lorentz metric to its associated derivative operator. Likewise, there is a second-order concomitant from $\mathcal{B}_{\mathrm{g}}$ to the geometric bundle $(\mathcal{B}_{\mathrm{Riem}}, \mathcal{M}, \pi_{\mathrm{Riem}}, \iota_{\mathrm{Riem}})$ of tensors with the same index structure and symmetries as the Riemann tensor, *viz.*, the mapping that takes each Lorentz metric to its associated Riemann tensor. (This is the precise sense in which the Riemann tensor associated with a given Lorentz metric is "a function of the metric and its partial derivatives up to second order".) It is easy to see, moreover, that both concomitants are homogeneous of degree 0.

It follows from theorem 6 and proposition 5 that a concomitant of the metric will be second order if and only if it is a zeroth-order concomitant of the Riemann tensor:

Proposition 7 *A concomitant of the metric is second-order if and only if it can be expressed as a sum of terms consisting of constants multiplied by the Riemann tensor, the Ricci tensor, the Ricci scalar curvature, and contractions and products of these with the metric itself.*

Now, in order to make precise the idea of having the physical dimension of stress-energy, recall that in general relativity all the fundamental units one uses to define stress-energy, namely time, length and mass, can themselves be defined using only the unit of time (or, equivalently, only the unit of length or mass); these are so-called geometrized units (Misner, Thorne, and Wheeler 1973, p. 36).[2] This guarantees that units of mass and length scale in precisely the same manner as the time-unit when new units of time are chosen by multiplying the time-unit by some fixed real number $\lambda^{-\frac{1}{2}}$. (The reason for the inverse square-root will become clear in a moment). Thus, a duration of t time-units would become $t\lambda^{-\frac{1}{2}}$ of the new units; an interval of d units of length would likewise become $d\lambda^{-\frac{1}{2}}$ in the new units, and m units of mass would become $m\lambda^{-\frac{1}{2}}$ of the new units. This justifies treating all three of these units as "the same", and so expressing acceleration, say, in inverse time-units. To multiply the length of all timelike vectors representing an interval of time by $\lambda^{-\frac{1}{2}}$, however, is equivalent to multiplying the metric by λ (and so the inverse metric by λ^{-1}), and indeed such a multiplication is the standard way one represents a change of units in general relativity. This makes physical sense as the way to capture the idea of physical dimension: all physical units, the ones composing the dimension of any physical quantity, are geometrized in general relativity in the most natural formulation, and so depend only on the scale of the metric itself. By Weyl's Theorem, however, a metric times a constant represents exactly the same physical phenomena as the original metric (Malament 2012, ch. 2, §1).[3]

2. Aldersley (1977) contains an interesting discussion of geometrized units, and proves a result superficially similar to theorem 2, albeit in a very different way than I give here. I have trouble understanding many of his arguments and conclusions, however, as he seems to imply that the physical dimensions of the components of a quantity depend on the physical dimensions of the coordinates in a coordinate system in which the quantity is represented. This makes no sense to me. A quantity simply has a physical dimension, and how one represents it in a coordinate system, if one does at all, is physically irrelevant to that fact. Moreover, his "Axiom of Dimensional Analysis" (p. 372), on which his arguments are based, seems to me similarly flawed, in that its statement depends on his claim that the physical dimensions of the components of a quantity can differ in different coordinate systems.

3. Recall that Weyl's Theorem states that the projective structure and the conformal structure determine the metric up to a constant.

Now, the proper dimension of a stress-energy tensor can be determined by the demand that the Einstein field-equation, $G_{ab} = 8\pi\gamma T_{ab}$, remain satisfied when one rescales the metric by a constant factor. γ has dimension $\frac{(\text{length})^3}{(\text{mass})(\text{time})^2}$, and so in geometrized units does not change under a constant rescaling of the metric. Thus T_{ab} ought to transform exactly as G_{ab} under a constant rescaling of the metric. A simple calculation shows that G_{ab} $(= R_{ab} - \frac{1}{2}Rg_{ab})$ remains unchanged under such a rescaling. Thus, a necessary condition for a tensor to represent stress-energy is that it remain unchanged under a constant rescaling of the metric. It follows that the concomitant at issue must be homogeneous of weight 0 in the metric, whatever order it may be.

We must still determine the order of the required concomitant. In fact, the weight of a homogeneous concomitant of the metric suffices to fix the differential order of that concomitant.[4] This can be seen as follows, as exemplified by the case of a two covariant-index, homogeneous concomitant S_{ab} of the metric. A simple calculation based on definition 4 and on the fact that the concomitant must be homogeneous shows that the value of an n^{th}-order concomitant S_{ab} at a point $p \in \mathcal{M}$ can be written in the general form

$$S_{ab} = \sum_{\alpha} k_\alpha \, g^{qx} \dots g^{xr} \left(\widetilde{\nabla}_x^{(n_1)} g_{qx} \right) \dots \left(\widetilde{\nabla}_x^{(n_i)} g_{xr} \right) \tag{1}$$

where: $\widetilde{\nabla}_a$ is any derivative operator at p *other* than the one naturally associated with g_{ab}; 'x' is a dummy abstract index; '$\widetilde{\nabla}_x^{(n_i)}$' stands for n_i iterations of that derivative operator (obviously each with a different abstract index); α takes its values in the set of all permutations of all sets of positive integers $\{n_1, \dots, n_i\}$ that sum to n, so i can range in value from 1 to n; the exponents of the derivative operators in each summand themselves take their values from α, *i.e.*, they are such that $n_1 + \cdots + n_i = n$ (which makes it an n^{th}-order concomitant); for each α, k_α is a constant; and there are just enough of the inverse metrics in each summand to contract all the covariant indices but a and b.

Now, a combinatorial calculation shows

Proposition 8 *If, for $n \geq 2$, S_{ab} is an n^{th}-order homogeneous concomitant of g_{ab}, then to rescale the metric by the constant real number λ multiplies S_{ab} by λ^{n-2}.*

In other words, the only such homogeneous n^{th}-order concomitants must be of weight $n - 2$.[5] So if one knew that S_{ab} were multiplied by, say, λ^4 when the metric was rescaled by λ, one would know that it had to be a sixth-order concomitant. In particular, S_{ab} does not rescale when $g_{ab} \to \lambda g_{ab}$ only if it is a second-order homogeneous concomitant of g_{ab}, *i.e.*, (by theorem 6 and proposition 7) a zeroth-order concomitant of the Riemann tensor. There follows from proposition 5

4. I thank Robert Geroch for pointing this out to me.

5. The exponent $(n - 2)$ in this result depends crucially on the fact that S_{ab} has only two indices, both covariant. One can generalize the result for tensor concomitants of the metric of any index structure. A slight variation of the argument, moreover, shows that there does not in general exist a homogeneous concomitant of a given differential order from a tensor of a given index structure to one of another structure—one may not be able to get the number and type of the indices right by contraction and tensor multiplication alone.

Lemma 9 *A 2-covariant index concomitant of the Riemann tensor is homogeneous of weight zero if and only if it is a zeroth-order concomitant.*

Thus, such a tensor has the physical dimension of stress-energy if and only if it is a zeroth-order concomitant of the Riemann tensor. It is striking how powerful the physically motivated assumption that the required object have the physical dimensions of stress-energy: it guarantees that the required object will be a second-order concomitant of the metric.

Now, it follows from proposition 7 that the only possibilities for geometrical objects to place on the lefthand side of a field equation that would play the role of the Einstein field equation are linear combinations of the Ricci tensor and the scalar curvature multiplied by the metric. The only covariantly divergence-free, linear combinations of those two quantities, however, are constant multiples of the Einstein tensor G_{ab}. (To see this, note that if there were another, say $k_1 R_{ab} + k_2 R g_{ab}$ for constants k_1 and k_2, then $k_1 R_{ab} + k_2 R g_{ab} - 2k_2 G_{ab}$ would also be divergence free, but that expression is just a constant multiple of the Ricci tensor.) This proves theorem 2. A benefit of the proof is that it gives real geometrical and physical insight into the result, insight not provided by Lovelock's original proof of theorem 1, which consists of several pages of unilluminating coordinate-based, brute-force calculation.

Theorem 2 shows the uniqueness of the Einstein field equation in all dimensions. The theorem is similar to Lovelock's result, but different in four important ways. The first difference is that I require the concomitant of the metric to be homogeneous of weight zero. The physical interpretation of this is that the desired tensor have the physical dimensions of stress-energy, as is the case for the Einstein tensor, and as must be the case for any tensor that one would equate to a material stress-energy tensor to formulate a field equation. This provides a physical interpretation to the conditions of the theorem that Lovelock's theorem lacks. It also leads to the second difference: one does not need to assume that the desired concomitant is second-order; that property falls naturally out of the proof.

The third difference is that the theorem holds in all dimensions, not just in four. In higher dimensions, there are other tensors satisfying Lovelock's original theorem, the so-called Lovelock tensors. (Those tensors are not linear in the second-order partial-derivatives of the metric as the Einstein tensor is.) Those tensors form the basis of so-called Lanczos-Lovelock gravity theories in dimensions higher than four (Lovelock 1971; Padmanabhan and Kothawala 2013), being used to formulate field equations including Lovelock tensors besides the Einstein tensor. Because theorem 2 holds in all dimensions, not just in four, it follows that, in dimensions other than four, the Lovelock tensors are not homogeneous of weight zero, and so do not have the physical dimension of stress-energy. Thus, if one wants to construct a field equation that equates a linear combination of such tensors to the stress-energy tensor of ordinary matter, as Lovelock theories of gravity do, then the coupling constants cannot be dimensionless like Newton's gravitational constant; the physical dimension of each coupling constant will be determined by the physical dimension of the Lovelock tensor it multiplies. These Lovelock tensors are usually interpreted as generalizing the Einstein field equation so as to include curvature terms other than the Einstein tensor that couple with the stress-energy of ponderable matter. As in the case of the cosmological constant, however, the fact that these Lovelock tensors require dimensionful coupling constants to get the physical dimensions of the terms right strongly suggests that one ought not interpret them as geometrical terms coupling to ordinary stress-energy, but rather as exotic forms of stress-energy themselves. If this is correct, then Lovelock theories are not in fact generalizations of general relativity, but rather simply the Einstein field equation with exotic stress-energy added to the righthand side.

The fourth difference is that the addition of constant multiples of the metric is not allowed. I interpret that to mean that any cosmological-constant term must be construed as part of the total stress-energy tensor of spacetime, and so, in particular, the cosmological constant itself must have the physical dimensions of $(\text{mass})^2$.

I conclude with two remarks. First, theorem 2 has another natural interpretation: it shows in a precise and rigorous sense the nonexistence of a gravitational stress-energy tensor. If there were such a thing, we would expect it to depend on curvature, and so be zero in and only in flat spacetimes. Constant multiples of the Einstein tensor, however, are not appropriate candidates for the representation of gravitational stress-energy: the Einstein tensor will be zero in a spacetime having a vanishing Ricci tensor but a non-trivial Weyl tensor; such spacetimes, however, can manifest phenomena, *e.g.*, pure gravitational radiation in the absence of ponderable matter, that one naturally wants to say possess gravitational energy in some (necessarily non-localized) form or other.[6] (See Curiel 2018 for extended discussion of this and other interpretational issues raised in this paper.)

Second, the derivation of the Einstein field equation in Padmanabhan (2010), based on thermodynamical arguments, is really just a special case of theorem 2 in disguise, as the Einstein tensor is the only appropriate covariantly divergence-free tensor having the units of stress-energy, as his proof requires. (The same holds true for the generalization of Padmanabhan's arguments to Lanczos-Lovelock gravity in Padmanabhan and Kothawala 2013.) Note, moreover, that Lovelock's original theorem does not suffice for Padmanabhan's needs, since it is crucial that the desired tensor have the right physical dimension.

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6. This argument, by the way, obviates the common criticism of the claim that gravitational stress-energy, if there were such a thing, ought to depend on the curvature. The critics point out that that would make gravitational stress-energy depend on second-order partial derivatives of the field potential, whereas all other known forms of stress-energy depend only on terms quadratic in the first partial derivatives of the field potential. It is, however, exactly second-order, homogeneous concomitants of the metric that possess terms quadratic in the first partials. The rule is that the order of a homogeneous concomitant is the sum of the exponents of the derivative operators when the concomitant is represented in the form of equation (1).

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