The Geometry of the Euler-Lagrange Equation

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1 The Intrinsic Geometry of the Tangent Bundle

In this paper, I give a novel construction and presentation of the intrinsic geometry of a generic tangent bundle, in the terms of which the Euler-Lagrange equation can be formulated in a geometric, illuminating way. I conclude by proving a result (theorem 3.2) that shows that, in a strong sense, not only must Lagrangian mechanics be formulated on tangent bundles (as opposed to Hamiltonian mechanics, which can be formulated on any symplectic manifold, whether diffeomorphic to a cotangent bundle or not), but moreover the intrinsic geometry of the Euler-Lagrange equation itself allows one to completely reconstruct the space on which one formulates it as a tangent bundle over a particular base space.

First, I fix some notation. I shall mostly use the abstract-index notation of Geroch, Newman and Penrose to designate indexed entities such as tensors. (See, *e.g.*, Wald 1984 for an exposition of it, and Penrose and Rindler 1984 for the complete mathematical theory.) In particular, indices in this notation do *not* designate the ordinal position of coordinates in a fixed coordinate system, nor, indeed, anything pertaining to any coordinate system, and so the indexed sign itself does not designate the numerical component of an object in a coordinate system but rather the geometrical object itself. I shall use the following convention to keep things straight: Roman letters as indices will signify tensors in the abstract-index notation and Greek letters will signify coordinate functions and tensorial components in coordinate systems. Also, because it will be useful to have a way to distinguish objects living on bundles from those living on their base spaces, I shall use lower-case indices to signify objects residing on a base space (or arbitrary manifold) and upper-case for those on bundle spaces. Thus, for example, 'ξ^A' may designate a vector field on a bundle space as represented in the abstract-index notation, and 'η_α' a component of a 1-form on the base space of a bundle, in a particular coordinate system. It is assumed that all manifolds are smooth, connected, paracompact and Hausdorff and all structures on them are smooth.

Given a $2n$-dimensional manifold $\mathcal{N}$, an *almost-tangent structure* is a smooth one-up, one-down index tensor field $J^a{}_b$ satisfying the following conditions:[1]

1. considered as a linear operator on the tangent planes of $\mathcal{N}$, $J^a{}_b$ has rank n everywhere
2. $J^a{}_n J^n{}_b = 0$

It is not difficult to see that, as a linear operator on tangent vectors, the image of $J^a{}_b$ image equals its kernel, an n-dimensional distribution on $\mathcal{N}$. If this distribution is integrable in the sense of Frobenius, $J^a{}_b$ itself is said to be integrable. A necessary and sufficient condition for this to hold is that the Nijenhuis tensor of $J^a{}_b$ identically vanish,

$$J^n{}_{[b}\nabla_{|n|}J^a{}_{c]} - J^a{}_n\nabla_{[b}J^n{}_{c]} = 0$$

(*Cf.* Nijenhuis (1951) and Lehmann-Lejeune (1964).) Note that the quantity to the left of the equal sign does not depend on choice of derivative operator and in fact does define a tensor. It is easy to see that an integrable almost-tangent structure induces a locally affine structure on each leaf of its subordinate foliation (Brickell and Clark 1974). The almost-tangent structure itself will be said to be *complete* if this locally affine structure is complete, in the sense that the natural flat affine connection associated with the locally affine structure is geodesically complete.

If $\mathcal{M}$ is an n-dimensional manifold, then its tangent bundle $T\mathcal{M}$ comes equipped with a canonical integrable, complete almost-tangent structure $J^A{}_B$ (hence the name), the image of which is precisely the set of vertical vectors on $T\mathcal{M}$ (*i.e.*, those tangent to the fibers). In order to construct it and show its geometrical meaning, it will be useful to characterize first a few other structures natural to tangent bundles. $T\mathcal{M}$ has a distinguished vector space of vector fields $\boldsymbol{\mathcal{V}}$, the *vertical vector fields*, those everywhere tangent to the fibers. $\boldsymbol{\mathcal{V}}$ defines an n-dimensional distribution, which is clearly integrable (in the sense of Frobenius)—the leaves of the associated foliation are just the bundle's fibers, the linear structure of which guarantees integrability. Thus $\boldsymbol{\mathcal{V}}$ is involutive, *i.e.*, closed under the action of the Lie bracket.

Now, given a curve $\gamma(u)$ on $\mathcal{M}$, there is a natural way to associate with it a curve $\hat{\gamma}(u)$ on $T\mathcal{M}$:

$$\hat{\gamma}(u) = (\gamma(u),\, \gamma^a(u))$$

where $\gamma^a(u)$ is the tangent vector to γ at the parameter value u. It is easy to see that a curve ξ on $T\mathcal{M}$, with tangent vector ξ^A, has such an associated curve on $\mathcal{M}$ if and only if

$$d\pi_T \circ \xi^A = \mathrm{Id}_{T\mathcal{M}} \tag{1.1}$$

where $d\pi_T$ is the differential of π_T (*i.e.*, a mapping from $TT\mathcal{M}$ to $T\mathcal{M}$); ξ^A is treated as a mapping from $T\mathcal{M}$ to $TT\mathcal{M}$; and $\mathrm{Id}_{T\mathcal{M}}$ is the identity map on $T\mathcal{M}$. I shall call a vector ξ^A on $T\mathcal{M}$ tangent

[1] See Clark and Bruckheimer (1960), Dombrowski (1962) and Eliopoulos (1962) for the original work on this structure.

to such a curve a *second-order vector field*.[2] It follows from condition (1.1) that the space $\boldsymbol{\Xi}$ of second-order vector fields has the structure of an affine space over $\boldsymbol{\mathcal{V}}$.

We are now in a position to construct the canonical almost-tangent structure on $T\mathcal{M}$, using the following fundamental lemma, proved by a simple computation, which I skip:[3]

Lemma 1.1 *Given any vector field α^A on $T\mathcal{M}$ and any second-order vector field ξ^A, there is a unique vertical vector field β^A such that*

$$\pounds_\xi \beta^A + \alpha^A \tag{1.2}$$

is a vertical vector field.

Fix for a moment the second-order vector field ξ^A in the expression (1.2), and treat β^A as a function of α^A. If we multiply α^A by a scalar field σ, then a simple computation shows that β^A gets multiplied by σ as well. Thus there is a tensor field $J^A{}_B$, depending on ξ^A, such that $J^A{}_N \alpha^N = \beta^A$. The dependence of $J^A{}_B$ on ξ^A, however, is in fact trivial: because every second-order vector field can be derived by adding a vertical vector field to ξ^a, lemma 1.1, in conjunction with the integrability of $\boldsymbol{\mathcal{V}}$, implies that $J^A{}_B$ is the same no matter which second-order vector field one uses to construct it. Now, the contravariant index of $J^A{}_B$ always lies in the vertical sub-space of the tangent plane, since β^A is always a vertical vector, and its covariant index must annihilate vertical vectors by uniqueness: if α^A is already vertical, then β^A must be 0. It follows that $J^A{}_N J^N{}_B = 0$. In fact, it follows that a vector α^A is vertical if and only if $J^A{}_N \alpha^N = 0$ and that every vertical vector is of the form $J^A{}_N \alpha^N$ for some α^A. Thus $J^A{}_B$ defines an isomorphism from the quotient space of the tangent plane by the vertical vectors to the space of vertical vectors at every point of $\mathcal{N}$. $J^A{}_B$ is therefore an almost-tangent structure. A useful property of $J^A{}_B$ is that

$$J^A{}_N \pounds_\xi J^N{}_B = -J^A{}_B \tag{1.3}$$

for any second-order vector field ξ^A. To see this, note that by definition $\pounds_\xi (J^A{}_N \alpha^N) + \alpha^A$ is always a vertical vector; equation (1.3) then follows from expanding the Lie derivative, contracting with $J^A{}_B$ and noting that α^A is arbitrary. Because $J^A{}_N J^N{}_B = 0$, one also has

$$J^N{}_B \pounds_\xi J^A{}_N = J^A{}_B$$

It follows from the affine-space structure of $\boldsymbol{\Xi}$ that $J^A{}_B$ annihilates the difference of any two second-order vector fields. Thus, for any $\xi^A \in \boldsymbol{\Xi}$, $\Lambda^A = J^A{}_N \xi^N$ is a vertical vector field independent of the choice of ξ^A. Λ^A is the *Liouville vector field*, a canonical vertical vector field on $T\mathcal{M}$. It follows from lemma 1.1 that $\pounds_\Lambda \xi^A$ is a second-order vector field if and only if ξ^A is. Thus $J^A{}_N \pounds_\Lambda \xi^N = \Lambda^A$.

[2] 'Second-order' because such vector fields naturally represent second-order differential equations on $\mathcal{M}$, just as sections of $T\mathcal{M}$ represent first-order differential equations. Second-order vectors are also called in the literature 'lifts' and 'semi-sprays'. I shall not use 'lift', for there are actually several natural ways to "lift" structure from $\mathcal{M}$ to $T\mathcal{M}$ (Yano and Ishihara 1973), so 'lift' by itself is ambiguous. 'Semi-spray' just strikes me as a little silly, not to mention uninformative.

[3] I thank R. Geroch (private communication) for pointing this out to me.

Expanding the lefthand side and noting that $\pounds_\Lambda(J^A{}_N\xi^N) = \pounds_\Lambda \Lambda^A = 0$, it follows that $\xi^N \pounds_\Lambda J^A{}_N = -\Lambda^A$. Since this holds for arbitrary second-order vector fields, it follows that

$$\pounds_\Lambda J^A{}_B = -J^A{}_B \tag{1.4}$$

In fact, equation (1.4) can be used to define Λ^A since it is the only vertical vector field that satisfies it, as a simple computation shows.

Writing out the components of these structures in natural coordinates on $T\mathcal{M}$ gives some insight into their geometry.[4] Let $q \in \mathcal{M}$ have coordinates (q^α), so that any point p in the fiber over q will have naturally induced coordinates $(q^\alpha,\, v^\beta)$. Then any second-order vector field ξ^A can be written in the form

$$\xi^A = v^\mu \left(\frac{\partial}{\partial q^\mu}\right)^A + \xi^\nu \left(\frac{\partial}{\partial v^\nu}\right)^A$$

where the ξ^ν are arbitrary (smooth) functions of q^α and v^α. In other words, in so far as a vector on the tangent bundle can be thought of as an infinitesimal change in the base-space directions, as it were, plus an infinitesimal change in the vertical directions, a second-order vector always has the infinitesimal change in the base-space directions equal to the velocity tangent to a curve on the base space heading in that direction with the associated rate of change. The canonical almost-tangent structure has the form

$$J^A{}_B = \left(\frac{\partial}{\partial v^\mu}\right)^A \otimes (dq^\mu)_B$$

If ξ^A is a vector in $T_p(T\mathcal{M})$ with components $(\rho^\alpha,\, \sigma^\beta)$, then $J^A{}_N\xi^N$ is a vertical vector with components $(0,\, \rho^\alpha)$, *i.e.*, the original "non-vertical" components get shoved over and become the new vertical components while the original vertical components are annihilated. Λ^A has the form

$$v^\mu \left(\frac{\partial}{\partial v^\mu}\right)^A$$

Thus, Λ^A in essence tells you "how far" from the zero-section you are in the fiber. It is also easy to verify that the 1-parameter group of diffeomorphisms it generates are precisely the homothetia of the fibers that multiply vectors tangent to $T\mathcal{M}$ by e^t, where t is the parameter of the diffeomorphism, as it sweeps them along its flow-curves.

2 The Euler-Lagrange Equation

Fix a Lagrangian $L:\, T\mathcal{M} \to \mathbb{R}$. In terms of the structures characterized in §1, the Euler-Lagrange equation becomes

$$\pounds_\xi(J^N{}_A \nabla_N L) - \nabla_A L = 0 \tag{2.1}$$

[4] A coordinate system (q_α) on $\mathcal{M}$ naturally induces one on $T\mathcal{M}$, *viz.*, $(q_\alpha,\, v_\beta)$, where the v_α represent vectors tangent to curves on $\mathcal{M}$ when those curves are parametrized in terms of the q_α—roughly speaking, $v_\alpha = \dot{q}_\alpha$. These natural coordinates are the generalization of $(\mathbf{x},\, \mathbf{v})$ (position and velocity) as used to parametrize the dynamical space of states of a Newtonian particle moving in $\mathbb{R}^3$.

where ξ^A, the *Lagrangian vector field*, is the vector field determined by L, *i.e.*, the solution to the Euler-Lagrange equation.[5] It is sometimes convenient to write the equation in an expanded, equivalent form,

$$2J^M{}_A\xi^N\nabla_{[M}(J^S{}_{N]}\nabla_S L) - J^M{}_A\nabla_M(\Lambda^N\nabla_N L - L) = 0 \tag{2.2}$$

I call $\nabla_{[A}(J^S{}_{B]}\nabla_S L)$ the *Lagrangian 2-form* associated with L. An L for which the 2-form is symplectic is called *regular*. Clearly, if this 2-form is symplectic, then the existence of a ξ^A satisfying the Euler-Lagrange equations is guaranteed. It also guarantees, moreover, that the solution ξ^A is indeed a second-order vector field. I shall give a proof of this result to allow the reader a chance to become better acquainted with these structures.

Proposition 2.1 *If $\nabla_{[A}(J^N{}_{B]}\nabla_N L)$ is symplectic, the unique solution ξ^A on $T\mathcal{M}$ to equation 2.2 is a second-order vector field.*

To start, note that

$$\begin{aligned}
2\Lambda^N\nabla_{[A}(J^N{}_{N]}\nabla_N L) &= 2\Lambda^N J^S{}_{[N}\nabla_{A]}\nabla_S L + 2\Lambda^N(\nabla_{[A}J^S{}_{N]})\nabla_S L\\
&= -\Lambda^N J^S{}_A\nabla_N\nabla_S L + \Lambda^N(\nabla_A J^S{}_N)\nabla_S L - \Lambda^N(\nabla_N J^S{}_A)\nabla_S L\\
&= -\Lambda^N J^S{}_A\nabla_N\nabla_S L - J^S{}_N(\nabla_A\Lambda^N)\nabla_S L - \Lambda^N(\nabla_N J^S{}_A)\nabla_S L\\
&= -\Lambda^N J^S{}_A\nabla_N\nabla_S L + J^S{}_A\nabla_S L - J^N{}_A(\nabla_N\Lambda^S)\nabla_S L\\
&\qquad\qquad \text{(by equation 1.4)}\\
&= -\Lambda^N J^S{}_A\nabla_N\nabla_S L + J^S{}_A\nabla_S L - J^N{}_A\nabla_N(\Lambda^S\nabla_S L)\\
&\quad + J^N{}_A\Lambda^S\nabla_N\nabla_S L\\
&= -J^N{}_A\nabla_N(\Lambda^S\nabla_S L - L)\\
&= -2J^N{}_A\xi^M\nabla_{[N}(J^N{}_{M]}\nabla_N L)
\end{aligned}$$

Now showing that

$$-2J^N{}_A\xi^M\nabla_{[N}(J^N{}_{M]}\nabla_N L) = 2J^N{}_m\xi^M\nabla_{[A}(J^N{}_{N]}\nabla_N L)$$

[5] A simple calculation shows that this equation, when expressed in natural coordinates, is equivalent to the standard coordinate-based form of the Euler-Lagrange equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

when ξ^A is second-order. (*Cf.* Klein (1962), Klein (1963) and de León and Rodrigues (1989).)

will prove the proposition, since, on the assumption that $\nabla_{[A}(J^N{}_{B]}\nabla_N L)$ is symplectic, one will be able immediately to conclude that $J^A{}_N \xi^N = \Lambda^A$.

$$\begin{aligned}
-J^N{}_A \xi^M \nabla_{[N}(J^N{}_{M]}\nabla_N L) &= -J^N{}_A \xi^M J^S{}_{[M}\nabla_{N]}\nabla_S L - J^N{}_A \xi^M \nabla_{[N} J^S{}_{M]}(\nabla_S L) \\
&= -J^N{}_m \xi^M J^S{}_{[A}\nabla_{N]}\nabla_S L - J^N{}_A \xi^M \nabla_{[N} J^S{}_{M]}(\nabla_S L) \\
&\qquad \text{(by symmetry between } N \text{ and } S\text{)} \\
&= -J^N{}_m \xi^M \nabla_{[N}(J^N{}_{A]}\nabla_N L) + J^N{}_m \xi^M \nabla_{[N} J^S{}_{A]}(\nabla_S L) \\
&\quad - J^N{}_A \xi^M \nabla_{[N} J^S{}_{M]}(\nabla_S L) \\
&= J^N{}_m \xi^M \nabla_{[A}(J^N{}_{N]}\nabla_N L) \\
&\quad + \xi^M \nabla_S L(J^N{}_m \nabla_{[N} J^S{}_{A]} - J^N{}_A \nabla_{[N} J^S{}_{M]}).
\end{aligned}$$

A short calculation shows that the second term (the part in parentheses) on the righthand sideof the last line is the Nijenhuis tensor of $J^A{}_B$, which identically vanishes since $J^A{}_B$ is integrable on $T\mathcal{M}$, completing the proof.

Even though one can in general find non-trivial second-order vector fields that are solutions to equation 2.2 when the Lagrangian 2-form is not symplectic, such solutions are not in general unique, and so not of physical interest. I shall restrict myself, therefore, in the following to Lagrangians that make this symplectic. Interestingly enough, these are precisely the ones for which, in a fixed coordinate system on $T\mathcal{M}$,

$$\det \left| \frac{\partial^2 L}{\partial v^\mu \, \partial v^\nu} \right| \neq 0$$

the necessary and sufficient condition for one to be able to pass to a regular, unconstrained Hamiltonian formulation of the system via the Legendre transform. Clearly, a necessary condition for this to hold is that the Lagrangian be at least quadratic in all velocity terms.

When ξ^A is not second-order, then of course $\nabla_{[A}(J^N{}_{B]}\nabla_N L)$ will not be symplectic, but one may still wonder whether in this case one can find non-trivial solutions to equation 2.2 that are also solutions to the ordinary Euler-Lagrange equation. It turns out one can if L satisfies a certain other condition. If ξ^A is not second-order, say with components $(g^\alpha,\, h^\beta)$ in natural coordinates $(q^\alpha,\, v^\beta)$, equation 2.2 becomes a pair of equations to be jointly satisfied, the ordinary Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

and

$$\sum_{\mu=1}^{n} (g^\mu - v^\mu)\, \frac{\partial^2 L}{\partial v^\mu \partial v^\nu} = 0$$

for each ν separately. Given that ξ^A is not second-order, this latter equation will be satisfied if and only if

$$\sum_{\mu=1}^{n} \frac{\partial^2 L}{\partial v^\mu \partial v^\nu} = 0 \tag{2.3}$$

for every $\nu = 1, \ldots, n$.[6] In this case, the pair of equations will reduce down to the ordinary Euler-Lagrange equation for which $\nabla_{[A}(J^N{}_{B]}\nabla_N L)$ is not symplectic, and there are in general many solutions to this equation for a given L.

To get a feel for condition (2.3), consider the case where $\mathcal{M}$ is two-dimensional; then, using x for v^1 and y for v^2, the condition becomes

$$\begin{aligned} \frac{\partial^2 L}{\partial x^2} &= -\frac{\partial^2 L}{\partial x \partial y} \\ &= \frac{\partial^2 L}{\partial y^2}. \end{aligned} \tag{2.4}$$

Changing coordinates to $s = x + y$ and $t = x - y$ yields the equivalent conditions

$$\frac{\partial^2 L}{\partial s \partial t} = 0$$

and

$$\frac{\partial^2 L}{\partial s^2} = 0$$

which clearly has the general solution

$$L = ks + f(t)$$

where k is a constant and $f(t)$ is any twice-differentiable function of t alone. Switching back to v^1 and v^2, we find

$$L = k_1 v^1 + k_2 v^2 + f(v^1 - v^2)$$

to be the general form of a Lagrangian satisfying 2.3 in two dimensions.

This procedure can be generalized to any dimension without too much trouble. If $\mathcal{M}$ is n-dimensional, define the matrix

$$M^\mu{}_\nu \equiv e^{\frac{2\pi i j}{n}}$$

Then the general form of a Lagrangian satisfying equation 2.3 is

$$L = k_1 v^1 + \cdots + k_N v^N + f(M^2{}_\nu v^\nu, \ldots, M^N{}_\nu v^\nu)$$

where the k_μ's are constants and f is a twice-differentiable function of the v^μ's alone, taking $n-1$ arguments.

Thus far I have treated only the homogeneous Euler-Lagrange equation. I turn now to examine the inhomogeneous case,

$$2J^M{}_A \xi^N \nabla_{[M}(J^S{}_{N]}\nabla_S L) - J^M{}_A \nabla_M(\Lambda^S \nabla_S L - L) = \phi_A$$

where ϕ_A, the *generalized force-form*, is a 1-form on $T\mathcal{M}$. Let us call a 1-form ω_A on $T\mathcal{M}$ *anti-vertical* if it annihilates vertical vectors.[7] It is easy to see that ω_A is anti-vertical if and only if $J^N{}_A \omega_N = 0$. An argument similar to that used to prove 2.1 proves

[6] Note that if L is such as to satisfy this sum, then $\nabla_{[A}(J^N{}_{B]}\nabla_N L)$ is automatically not symplectic, as one should have expected. The sum says that adding the elements of each row separately of the matrix $\frac{\partial^2 L}{\partial v^\mu \partial v^\nu}$ yields zero, implying that the sum of the columns of the matrix add to zero. Thus the columns are not linearly independent and so the determinant of the matrix vanishes, which is to say that $\nabla_{[A}(J^N{}_{B]}\nabla_N L)$ is not symplectic.

[7] Such 1-forms are often called 'vertical' in the literature, but I find that confusing.

Proposition 2.2 *If the Lagrangian 2-form in the inhomogeneous Euler-Lagrange equation is symplectic, then the Lagrangian vector field is second-order if and only if the force-form ϕ_A is anti-vertical.*

When $\phi_A = 0$, simple inspection of equation 2.2 shows that the scalar function

$$E_L \equiv \Lambda^N \nabla_N L - L$$

is conserved along the flow-lines of L's Lagrangian vector field ξ^A, *i.e.*,

$$\pounds_\xi E_L = 0$$

By Poincaré's lemma, we know that locally there exists a scalar function U on $T\mathcal{M}$ such that

$$\phi_A = -\nabla_A U$$

if and only if

$$\nabla_{[A}\phi_{B]} = 0$$

In this case, one says that ϕ_A is *conservative*. One can easily show that

$$\pounds_\xi (E_L + U) = 0$$

in this case. Thus $E_L + U$ is identified with the total energy of the system.

If $\nabla_{[A}\phi_{B]} \neq 0$, then one says the dynamical system is *non-conservative*. One can still in this case define a notion of work done along any given evolution curve of the system. Let ξ be an evolution curve associated with the Lagrangian vector field ξ^A, that is, a solution to the inhomogeneous Euler-Lagrange equation with generalized force-form ϕ_A; then define W, *the total work done along ξ by the system*, by

$$W[\xi] = \int_\xi \xi^N \phi_A \, dt$$

When $\nabla_{[A}\phi_{B]} = 0$, this definition implies conservation of total energy, as it ought.

3 Lagrangian Mechanics and Tangent Bundle Structure

In order to write down the Euler-Lagrange equation on the tangent bundle, one requires $J^A{}_B$ and (at least implicitly) Λ^A. All of the arguments one uses in proving the important results so far obtained, such as propositions 2.1, ultimately rely only on the fact that $J^A{}_B$ is an integrable, complete almost tangent structure, that $J^A{}_N \Lambda^N = 0$, and that $\pounds_\Lambda J^A{}_B = -J^A{}_B$. This raises the question whether one could "do" Lagrangian mechanics on a $2n$-dimensional manifold with an integrable, complete almost-tangent structure and an appropriate Liouville-like vector field, but that was not diffeomorphic to the tangent bundle of any manifold, as one can "do" Hamiltonian mechanics on any symplectic manifold whether it is diffeomorphic to a cotangent bundle or not.

The following theorem answers the question:

Theorem 3.1 (Brickell and Clark, 1974) *If $\mathcal{N}$ is a manifold with an integrable, complete almost-tangent structure $J^A{}_B$ and a global vector field Λ^A satisfying $J^A{}_N \Lambda^N = 0$ and $\pounds_\Lambda J^A{}_B = -J^A{}_B$, then $\mathcal{N}$ is diffeomorphic to the tangent bundle of some manifold.*

$J^A{}_B$ and Λ^A by themselves encode all the tangent-bundle structure of $T\mathcal{M}$. If one were given $T\mathcal{M}$ simply as a differentiable manifold with $J^A{}_B$ and Λ^A defined on it, one could reconstruct the closed submanifolds of $T\mathcal{M}$ corresponding to its fibers; one could recover the vector space structure of each fiber; and one could recover $\mathcal{M}$ up to diffeomorphism. Roughly speaking, the leaves of the foliation induced by $J^A{}_B$ are the fibers as affine spaces (since $J^A{}_B$ tells you the vertical vectors, and fixes the non-vertical vectors up to the addition of a vertical vector), and Λ^A fixes the zero-section, *i.e.*, the origin of each such "affine-space fiber". $\mathcal{M}$ is then diffeomorphic to the quotient space of $T\mathcal{M}$ under the equivalence relation 'belongs to the same leaf of the $J^A{}_B$-foliation as'. Hence one may as well always do Lagrangian mechanics on the tangent bundle of configuration space. Contrast this with the situation in Hamiltonian mechanics, wherein the canonical symplectic structure does not by itself suffice to fix the structure of phase space as the cotangent bundle of configuration space.

Next, then, one naturally wants to know how much the structure of Lagrangian mechanics by itself, that is, the way that vector fields get associated with scalar fields, determines the structure of velocity-phase space as a tangent bundle. Roughly speaking, if one knew of the space $T\mathcal{M}$ merely as a differentiable manifold (*i.e.*, one did not know that it was the tangent bundle of $\mathcal{M}$), and one also knew the Lagrangian dynamical vector field associated with any given Lagrangian—say one had a black box that spat out the correct Lagrangian vector field on $T\mathcal{M}$ 3 seconds after one fed a Lagrangian into it—would this information alone suffice to reconstruct $J^A{}_B$ and Λ^A on $T\mathcal{M}$? That is, would this alone suffice to determine not only that $T\mathcal{M}$ was the tangent bundle of some manifold or other but actually to produce the manifold of which it was the tangent bundle (up to diffeomorphism)? The answer is yes. More precisely:

Theorem 3.2 *Given two n-dimensional manifolds $\mathcal{M}$ and $\hat{\mathcal{M}}$, and a diffeomorphism $\zeta : T\mathcal{M} \to T\hat{\mathcal{M}}$ satisfying the following condition:*

> *for any regular Lagrangian L on $T\mathcal{M}$ and its Lagrangian vector field ξ^A,*
>
> $$(\zeta^*)\,[\xi^A] = \hat{\xi}^A$$
>
> *where $\hat{\xi}^A$ is the Lagrangian vector field on $T\hat{\mathcal{M}}$ associated with $\hat{L} \equiv (\zeta^*)\,[L]$, where (ζ^*) is the natural push-forward action associated with ζ*

then there exists a diffeomorphism $z : \mathcal{M} \to \hat{\mathcal{M}}$ such that ζ is the diffeomorphism between $T\mathcal{M}$ and $T\hat{\mathcal{M}}$ naturally induced by the action of z.

To prove the theorem, it will be convenient to use the following

Lemma 3.3 *If ζ is a diffeomorphism from $T\mathcal{M}$ to $T\hat{\mathcal{M}}$ that takes second-order vector fields to second-order vector fields, then ζ arises from a diffeomorphism between $\mathcal{M}$ and $\hat{\mathcal{M}}$.*

Because the almost-tangent structure and the Liouville vector field determine the base manifold up to diffeomorphism, it will suffice to show that the push-forward action of ζ preserves them. Let ξ^A and $\hat{\xi}^A$ be second-order vector fields on $\mathcal{M}$ and $\hat{\mathcal{M}}$ respectively such that

$$(\zeta^*)[\xi^A] = \hat{\xi}^A$$

Then

$$J^A{}_B \xi^b = \Lambda^A$$

and

$$\hat{J}^A{}_B \hat{\xi}^b = \hat{\Lambda}^A$$

By hypothesis, therefore,

$$\hat{J}^A{}_B (\zeta^*)[\xi^A] = \hat{\Lambda}^A$$

Because ζ is a diffeomorphism we also have

$$(\zeta^*)[J^A{}_B \xi^b] = (\zeta^*)[\Lambda^A]$$

We can immediately conclude (up to a constant factor we may as well set equal to 1)

$$\begin{aligned} (\zeta^*)[J^A{}_B] &= \hat{J}^A{}_B \\ (\zeta^*)[\Lambda^A] &= \hat{\Lambda}^A \end{aligned} \tag{3.1}$$

proving the lemma.

Now let $\mathcal{M}$, $\hat{\mathcal{M}}$ and ζ be as in the statement of the theorem, and ξ^A a second-order vector field on $T\mathcal{M}$. Then at any given point there exists a (non-unique) Lagrangian L such that ξ^A is the Lagrangian vector field of L in a neighborhood of that point. By hypothesis, $\hat{\xi}^A$ such that

$$(\zeta^*)[\xi^A] = \hat{\xi}^A$$

is a solution to the Euler-Lagrange equation on $T\hat{\mathcal{M}}$ with Lagrangian $(\zeta^*)[L]$. Thus $\hat{\xi}^A$ is a second-order vector field on $T\hat{\mathcal{M}}$, which proves the theorem.

The theorem essentially says that, if a diffeomorphism between two tangent bundles "commutes" with the Euler-Lagrange equation, then the diffeomorphism arises from one between the base spaces of the bundles. The intrinsic geometry of the Euler-Lagrange equation requires that spaces on which the equation can be meaningfully formulated be canonically diffeomorphic to tangent bundles, and, moreover, that intrinsic geometry allows one to fully recover the geometry of the space on which one formulates the equation *as* a tangent bundle in a canonical way.

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