“Definability,” “Conventionality,” and Simultaneity in Einstein–Minkowski Space-Time

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For Abner Shimony—in gratitude for the light and warmth I have received from a lifelong friendship.

Abstract In this article, I attempt to clarify certain misunderstandings that have contributed to continuing controversy over the status of the concept of relative simultaneity in the special theory of relativity. I also correct a number of technical errors in the literature of the subject, and present several new technical results that may further serve to clarify matters.

Controversy over the status of the concept of relative simultaneity in the special theory of relativity has proved remarkably durable. Very recently (within two days of first writing these words), as a result of ruminations on a recent paper of Adolf Grünbaum’s, I have come to believe that an important contributing factor to the persistence of the dispute is the use of certain key words (or phrases) in quite different senses by some of the disputants. One central aim of this paper, therefore, is to (try to) clarify these misunderstandings, and thereby both to reduce the number of the points of disagreement, and for the remaining points—for one can hardly expect all disagreement to be thus dispelled—at least to help clarify what the disagreements really *are*. A second aim is to correct some technical errors in the literature of the subject, and to state and prove some new technical results that may help contribute to clarity in the matter.

1 David Malament’s Contribution: (a) Remarks on Some Technical Objections; (b) Refinements of the Theorem

In a recent “revisiting” of David Malament’s well-known discussion of this subject [1], Mark Hogarth [2, p. 492] writes as follows, quoting a remark of mine from [3]:

> Just how decisive is Malament’s result for the issue of conventionality of simultaneity? Howard Stein echoes a common sentiment:

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> The issue … has been dealt with—in my opinion, conclusively—by David Malament, who pointed out that the Einstein–Minkowski conception of relative simultaneity is not only characterizable in a direct geometrical way within the framework of Minkowski's geometry … but is the only possible such conception that satisfies certain very weak 'natural' constraints.

Hogarth then notes that Malament has made use of the assumption that change of scale is an automorphism of the structure of space-time; objects that physics itself is *not* invariant under change of scale; concludes that Malament's result is after all inconclusive; and offers an argument of his own to show that invariance under change of scale can be replaced by another requirement, and the uniqueness of the standard conception of simultaneity thereby rescued.[1]

In correspondence before his paper was published, I remarked to Hogarth that there is a footnote to the statement he quotes from me—see [3, p. 153, n. 1]—in view of which the statement cannot reasonably be interpreted to mean that—in my view—Malament "had, as it were, dotted all the i's and crossed all the t's of the subject"; that, rather, what I had meant was "that Malament had redirected the discussion, away from the consideration of alternative ways of introducing 'a time-coordinate,' to the consideration of what purely geometric notions are available in Minkowski geometry." The footnote in question reads:

> There is one slightly delicate point to be noted: Malament's discussion, which is concerned with certain views of Grünbaum, follows the latter in treating space-time without a distinguished time-orientation. To obtain Malament's conclusion for the (stronger) structure of space-time with a time-orientation, one has to strengthen the constraints he imposes on the relation of simultaneity: it suffices, for instance, to make that relation (as in the text above) relative to a *state of motion* (i.e., a time-like *direction*), rather than—as in Malament's paper—to an *inertial observer* (i.e., a time-like *line*).

I did not think it necessary to *demonstrate* the stated fact, taking it for granted that anyone who cared to would easily see how a proof would go.

Some years after [3] appeared, a paper was published by Sahotra Sarkar and John Stachel [4]—under the title, "Did Malament Prove the Non-Conventionality of Simultaneity in the Special Theory of Relativity?" These authors criticize Malament's argument on the very grounds mentioned in the footnote; but they elaborate upon these grounds in a way that in my opinion is very defective, and is in serious

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[1] Malament, requiring invariance under change of scale, adds, besides the condition of invariance under automorphisms of space-time, only the assumption that simultaneity relative to O (a) is an equivalence relation, and (b) holds for *at least one* pair of points (p, q) with p on O and q not on O, but does not hold for *every* pair of points.—Hogarth appeals to quantum field theory for the fact that physics is not invariant under change of scale (this is indeed already clear in classical physics, since the fundamental classical physical constants allow us, in more than one way, to *determine a unit of length*; moreover, at the very beginning of the modern science of physics, Day 1 of Galileo's *Two New Sciences* opens with this paradox: a scale model of, for instance, a ship, may be perfectly stable, but the ship built from this model may collapse under its own weight—and Galileo's spokesman Salviati says that, although geometry is invariant under change of scale, this *non-* scale-invariance of "machines" can be *explained* by geometry).

Hogarth, renouncing appeal to scale-invariance, strengthens Malament's assumption (b) by requiring that, for any "inertial observer world-line" O and any point p in space-time, there is one and only one point q on O to which p is simultaneous relative to O. (See also Supplementary Note 1.)

need of clarification. They also offer a proof of a result akin to that indicated in the footnote; but their proof is fallacious, and the result therefore *does* demand the explicit proof that I thought superfluous. Since Hogarth's theorem is subject to the same objection raised by Sarkar and Stachel—Hogarth, too, makes crucial use of reflection-invariance—it is clearly desirable to establish a conclusion that depends *neither* upon invariance under scale-change *nor* upon invariance under reflections.[2]

But first I wish to make it clear that Malament's theorem itself—exactly as he has formulated it—is entirely correct. This seems important to emphasize, because Sarkar and Stachel have challenged the entire correctness of this theorem.[3] A discussion of the point is especially desirable because it concerns the question, just what is meant when one says that a notion is "definable *from*" something or other—I shall say, definable from the elements of *a given kind of structure*, or from the basic notions of *a given mathematical theory* (understood as being "the theory of that kind of structure"); and it turns out that a misunderstanding on this score is also relevant to points raised by Adolf Grünbaum, which we shall consider later.

Malament's theorem explicitly refers to a relation "definable from κ and O," where κ is the binary relation, on the set of points of Minkowski space-time, "p and q are such that one of them may causally influence the other," and where O is a *given* straight time-like world-line (what I shall henceforth refer to—and have already referred to, in Supplementary Note 1—as an "observer-line"). (So strictly speaking, if the phrasing of the end of the preceding paragraph is taken pedantically, the "theory" concerned is "the theory of the relation κ in a Minkowski space-time with a particular observer-line O singled out.") Malament then says [1, p. 297],"If an n-place relation is definable from κ and O, in any sense of 'definable' no matter how weak, then it will certainly be preserved under all O causal automorphisms [that is: mappings that preserve both O and κ]." It is this statement that Sarkar and Stachel challenge: they acknowledge the correctness of Malament's theorem *if "causal definability"* (as they put it) *is construed in Malament's way*; but they deny that this is an appropriate way to construe such definability—and they give an alleged counterexample.

Now, the issue this raises is simply one of *logic*. And the logical situation should be altogether clear: for the logician—or the mathematician—to *define* a notion in terms of certain basic concepts is in effect to introduce an abbreviated mode of expression; any statement phrased using the defined notion may be rephrased using only those basic concepts: the "new" notion is simply eliminable. But that a statement using only the basic notions of a theory is unaffected by automorphisms of the

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[2] Sarkar and Stachel mention in passing [4, p. 215 n. 11, and p. 217] that they do not use scale-invariance in their proof; but, as remarked—and as will be shown below—the proof is invalid, so this fact is irrelevant.

[3] If this formulation seems odd, it should—it is designedly so; for although Sarkar and Stachel say, "Clearly, something is amiss with Malament's theorem," they add, "A *correct* mathematical result [emphasis added] seems to be contradicted by patently good counterexamples"; what they challenge is, not the soundness of Malament's argument, but "the interpretation of one of the conditions that Malament imposes on simultaneity relations" [4, p. 214]. So the result is "correct"—but not *entirely* so.

object treated of by the theory is an immediate consequence of the very definition of "automorphism." So "in principle," it is hard to see how a controversy can arise here.

"In practice," however, one sees how confusion *did* occur. Sarkar and Stachel point to the facts that "the distinction between the [two, oppositely directed,][4] half null cones of [the full null cone of an event e] can be made using causal definability alone" [4, p. 213], and that this distinction can be made "coherently" for all point-events of the Minkowski space-time concerned. They conclude that the following two relations, satisfying Malament's other criteria, can be defined from the null-cone structure of the space-time, and thus from the relation κ alone (O plays no role): (a) p lies on the mantle of the backwards null-cone at q; (b) p lies on the mantle of the forwards null-cone at q; and conclude therefore that both (a) and (b), applied to point-pairs (p,q) with q on O, constitute relations definable from κ and O that satisfy all Malament's other conditions on a simultaneity relation. These examples, they accordingly maintain, show that his theorem is incorrect when "definable from κ and O" is rightly construed.

The error here turns on an ambiguity in the notion of "distinguishing" two things—that is, "telling the difference between" them. A Minkowski space-time has two possible "time-orientations." This statement certainly implies that one can "tell them apart"—enough, at least, to count them. And yet, one can't "tell which is which"—the two time-orientations are like two bosons in this respect.—Well, to continue with amusing word-games in the vernacular (even the vernacular of physics), would soon grow tiresome, and might amplify confusion; that is why logical pedantry has its legitimate place: One can define from κ and O a set of relations, each of which satisfies Malament's other requirements. The set I mean[5] contains *two* relations—the two described by Sarkar and Stachel. But one *cannot* "distinguish"—"single out"—*either one* of those two relations in distinction from the other; that is, one cannot do this *in terms of* κ *and* O *alone*. If only one could *once* "simply point to" one of the two half null-cones at *one* space-time point, the structure of κ would allow this choice to be "spread around" all through space-time, and we should have one of Sarkar and Stachel's examples; and why, indeed, should one *not* be able to do so?—But that isn't the question: the question is, What is *definable* from κ and O? And Malament's answer stands.

There remains the point that this may not be "the right question"; and I think this may be argued from two different points of view. It *could*—it seems to me—be claimed quite plausibly by an adherent of a "causal theory of time," Grünbaum for instance, that such a view of time is not restricted to what can be characterized in terms of the *symmetric* relation κ—that, rather, the *non-symmetric* relation "causal influence may be propagated from p to q" is part of the basic apparatus of that

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[4] Sarkar and Stachel say "backwards and forwards"; but they emphasize—rightly—that these adjectives are mere labels, not to be taken as denoting the "past" and the "future"; so I have preferred to substitute a neutral characterization, lest the reader suppose that they have overstepped at this point.

[5] This—a set that bears upon the Sarkar-Stachel examples—is by no means the *only* set of relations (besides the one relation of Malament's theorem) that can be defined from κ and O—there are infinitely many others; but there is no need to consider them here.

theory. (I am not sure whether Grünbaum himself would take this position; he has not, so far as I know, actually done so; and we shall see presently that his real disagreement with Malament has an entirely different basis.) And it could—*can*—also be based, as in my remark in the cited footnote and in Sarkar and Stachel's further discussion [4, pp. 214 ff.], on the consideration that time-reversal (as well as reversal of spatial orientation, and change of scale) cannot be considered as clearly demanded by physics.[6]

Turning now to the theorem Sarkar and Stachel claim to prove that covers the vulnerable points in that of Malament, their formulation reads as follows [4, p. 216]:

> Standard simultaneity is the only non-vacuous simultaneity relation causally' definable from κ and O that depends only on an inertial frame, and not on the particular world line O initially chosen to define that inertial frame.

By a "non-vacuous simultaneity relation" is here meant a relation S that satisfies the following three conditions, borrowed from Malament: (1) S is an equivalence relation; (2) there exists a point p on O and a point q not on O such that $S(p,q)$; (3) S does not hold for *every* pair of space-time points. By "the inertial frame defined by a (straight, time-like) world-line O" is meant (of course) the family of all lines parallel to O (I shall also use the term "inertial system" for such a family). And by "causally' definable" is meant: invariant under all automorphisms of the Minkowski space-time that preserve the inertial frame and that are continuously connectible to the identity ("causal' automorphisms," as Sarkar and Stachel call them). This class of mappings excludes time-reversals and "spatial" reflections—it is precisely the class of all those automorphisms (in the standard sense) that preserve both time-orientation and the orientation of the whole manifold. In particular, *it includes changes of scale*.

The last point deserves to be emphasized. *With* it, the theorem is true. But Sarkar and Stachel say—rightly—that they actually do not make any use of scale-invariance in their proof; and *without* the assumption of scale-invariance, the theorem becomes *false*. There are infinitely many counterexamples, of which the simplest is this: Let $S(p,q)$ be the relation: "p and q belong to hyperplanes orthogonal to the direction of O, having, for some integer n, the orthogonal (time-like) Minkowski distance $n\tau$, where τ is a particular distance (given once for all)".[7] This relation clearly satisfies the conditions on a "non-vacuous simultaneity relation,"

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[6] I should not wish to be taken as endorsing every aspect of the critical discussion of this point in [4], but there is no need to argue the matter here: the general point, as I have here stated it, suffices for our purposes.—Let me add that the fact that invariance under change of scale might be challenged was likewise suggested in [3]—see p. 149, near the top—and that, accordingly (see p. 250), an alternative proof was indicated for the theorem demonstrated on p. 249, avoiding appeal to scale-invariance. Having mentioned this I must add to the criticisms I have made of passages by others, one directed to a passage of my own: the argument [3. p. 149] leading up to the theorem mentioned is sound, but the formulation of the theorem itself does *not* state correctly what the argument has proved: the theorem as there stated is *false*!—On this, see Supplementary Note 2 below.

[7] The other counterexamples are fairly obvious variants of this one, taking the coefficients of τ to be, for instance, rational numbers, or elements of a given real algebraic number-field, or of any given proper subfield of the real numbers, or indeed of any given additive proper subgroup of the real numbers.

and as clearly is invariant under all automorphisms (reflections too!) that preserve the inertial frame to which O belongs, *except* scale-invariance.

If one wants to avoid reliance on scale-invariance, then, Malament's conditions on the relation S must be strengthened. (This is quite acceptable: Malament has simply chosen a very weak set of conditions, not for any "ideological" reason, but simply to show how restricted the class of "causally definable" relations in his sense *is*.) A suitable condition is this: that, for a given point p, $S(p,q)$ holds for exactly one point q on each line of the given inertial system. But a far weaker condition suffices: namely, that (a) for at least one observer-line O of the given inertial system, no two distinct points p, q on O satisfy $S(p,q)$, and (b) at least one pair of distinct points (p,q) in space-time does satisfy $S(p,q)$. (It is obvious, just from invariance under translations—which *ipso facto* take the inertial system to itself—that condition (a) entails that no two distinct points on *any* line of the system satisfy S.) To forestall any possible ambiguity of terminology, I shall henceforth use the word "automorphism" to refer to the full class of mappings normally considered by mathematicians to be automorphisms of a Minkowski space-time (including, therefore, all reflections and changes of scale); these coincide with the "causal automorphisms" of Malament; I shall use the phrase "proper automorphisms" for those called "causal' automorphisms" by Sarkar and Stachel (these still include changes of scale, but do not include reflections of any sort, "spatial" or "temporal"); and I shall use the phrase "strict automorphisms" for proper automorphisms that preserve the scale.

Let us proceed to the proof of two theorems: the one stated by Sarkar and Stachel for a "non-vacuous" simultaneity relation and requiring invariance under change of scale, and one using our strengthened condition on the relation but not requiring scale-change invariance (both, however, relativize simultaneity to an inertial system, not just to a "single observer"). For convenience, the two theorems will be formulated as a single one with two "cases":

Theorem 1. *In a Minkowski space-time of three or more dimensions let there be singled out an inertial system* **I**, *and let S be an equivalence relation on the space-time points satisfying one of the following two sets of conditions:*[8]

(a)(1) *S is invariant under all strict automorphisms of the space-time that preserve* **I**;

(2) *S(p,q) does not hold if p and q are distinct points on a single world-line of* **I**;

(3) *S(p,q) holds for at least one pair of distinct points;*

(b)(1) *S is invariant under all proper automorphisms of the space-time that preserve* **I**;

(2) *S(p,q) does not hold for every pair of points;*

(3) *S(p,q) holds for at least one pair of points not on the same world-line of* **I**;

—then for any distinct points p, q, S(p,q) holds if and only if the line pq is orthogonal to the lines of **I**.

[8] $S(p,q)$ will also be expressed by saying "p and q are simultaneous for **I**," or "p and q are **I**-simultaneous"; and that a line, or vector, or direction, is perpendicular to the lines of **I** will also be expressed by saying that it is "perpendicular—or orthogonal—to **I**."

Proof. Let p, p', be any points on different lines of **I**—O and O' respectively—that are simultaneous for **I** (such points exist in both cases, (a) and (b)); let s be the orthogonal (space-like) distance of p' from O; and let M be a three-dimensional subspace of our space-time that contains the plane spanned by the parallel time-like lines O and O'. Under all rotations of M about O as axis, the images of the point p' are all the points of the circle through p', in the (space-like) plane of M orthogonal to **I**, having its center at the intersection of that plane with O. Since each such rotation can be extended to a strict automorphism of the entire space-time preserving **I** and leaving p fixed (just allow it to act as the identity on any orthogonal complement to M in the space-time), all the points of that circle are simultaneous with p, and therefore also with one another; and the vectors $p'p''$ from p' to the other points of that circle (i) are all orthogonal to **I** and (ii) have lengths that fill the half-open interval of real numbers $(0, 2s]$. We therefore have established the following.

Lemma 1. *If p' has space-like orthogonal distance s from the* **I***-line through a point p with which it is* **I***-simultaneous, then for every real number s'' with $0 < s'' \leq 2s$ there is a point p'',* **I***-simultaneous with p', such that (i) $p'p''$ is orthogonal to* **I** *and (ii) the space-like length of $p'p''$ is s''. (*This holds, be it also noted, for both cases, (a) and (b).*)*

But this has the following as an almost immediate corollary: *Any two distinct space-time points q, q' such that the vector qq' is orthogonal to* **I** *are* **I***-simultaneous.*—Indeed, for our special point p' above, the lemma shows that the property possessed by the real number s is also possessed by (among others) $2s$ (for if p_1 is the point diametrically opposite p' in the circle, the space-like orthogonal distance of p' from the **I**-line through p_1 is just the space-like distance of p' from p_1 itself, namely $2s$); therefore, by successive doubling, that property is possessed by arbitrarily large real numbers;[9] and then, by the full conclusion of the lemma, it is possessed by every positive real number smaller than some "arbitrarily large" one—i.e., by all real numbers. Now let r be the space-like distance between our points q *and* q'. There is a point p'', by what has already been shown, **I**-simultaneous with p', such that $p'p''$ is orthogonal to **I** and has space-like length r. The translation taking p' to q is a strict automorphism of the space-time that preserves **I**, so the image q'' of p'' by that translation is **I**-simultaneous with q, and the vector qq'' is orthogonal to **I** and has space-like length r. Moreover, there is a rotation that leaves q fixed, preserves **I**, and takes q'' to q'; therefore q' is **I**-simultaneous with q.

We now know that any two points satisfying the "standard" condition for simultaneity relative to **I** are **I**-simultaneous. But to establish the converse is trivial: For case (a), we have only to remark that if Σ is a hyperplane orthogonal to **I**, p any point of Σ, and q any point **I**-simultaneous with p, the line of **I** through q meets Σ in a point q' that is **I**-simultaneous with q by the preceding result, and therefore is also simultaneous with q. So in case (a), q' must coincide with q—i.e., q must belong to Σ—since we cannot have distinct **I**-simultaneous points on a time-like line. As for case (b), the argument just given shows that if Σ is not the complete class of points

[9] Formally, an argument by mathematical induction is of course required.

I-simultaneous with p, there must be two distinct points q, q', that belong to the same line of **I** and that are **I**-simultaneous with p and therefore with one another. By changes of scale, the vectors qq' and $q'q$ can be transformed to vectors of arbitrary time-like length, and of both senses, all pointing in the direction of **I**. But such vectors, starting from all the points of Σ, reach all the points of the space-time; and since their end-points are simultaneous, we shall have that every point of the space-time is **I**-simultaneous with a point of Σ; so, since all the points of Σ are themselves **I**-simultaneous, all the points of space-time will be **I**-simultaneous, contradicting (b)(2): the proof is complete.[10]

The main point of these results is, as I have intimated, rather obvious; if the proofs are a bit lengthy and a little intricate, that is the result of making the conditions posited very Spartan. Thinking of the matter more broadly, it is easy to see—and to prove—that a relation, satisfying conditions one would surely ask of simultaneity, that is invariant under (a) every rotation about a line of the inertial system **I** and (b) every translation of space-time can have no other equivalence-classes than the hyperplanes orthogonal to the lines of **I**. Relativizing simultaneity to an inertial system is of course in consonance with the original procedure of Einstein, who envisaged, as coordinating their spatio-temporal observations, a "community of observers," in a shared "inertial state." But it has occurred to me to ask whether one can reach any sort of result from a weaker assumption—one that does not require *ab initio* that the "observers" of this community be at rest relative to one another; and indeed—in a certain sense—one can. I do not think that the results I shall now present are of great *philosophical* interest (this I shall discuss later); but I think they are of some—although again I should not say of "great"—*mathematical* interest; they are *not* "obvious."

Let us, then, make the following assumptions, for a Minkowski space-time of at least three dimensions (as we shall see, the two-dimensional case is notably different):

With each observer-line O there is associated an equivalence relation S_O on the whole space-time, in such a way that:

($S1$) any strict automorphism that maps O to O' transforms S_O to $S_{O'}$;
($S2$) if Σ is an equivalence-class of S_O then:

(a) every observer-line O' has one and only one point in common with Σ, and
(b) for every point p' of Σ there is an observer-line O' containing p' such that Σ is an equivalence-class of $S_{O'}$ (as well as of S_O).

—From these assumptions it does *not* follow that S_O is the standard Einstein–Minkowski[11] relation; let us examine what does follow.

[10] The counterexamples already given to the Sarkar-Stachel theorem with scale-changes excluded make it plain that the inference from the assumption that an **I**-line contains two simultaneous points to the conclusion that all its points are simultaneous could not be made without the condition of scale-change invariance. (For further discussion, locating the particular fallacy in the Sarkar–Stachel proof, see Supplementary Note 3.)

[11] I continue to use this designation, rather than "Poincaré–Einstein" as in [4], because I think it historically far more justified. Poincaré's treatment of what we now call "the special theory of

Let Σ be the equivalence-class of the relation S_O (for a given O), let p be the point of O in Σ, and let p' be any other point of Σ (note that p, as well as p', is an *arbitrary* point of Σ—for, by $(S2)$(b), *every* point of Σ is the point, in Σ, of *some* observer-line having Σ as an equivalence-class). A key lemma in our discussion will be this: *If O' is the observer-line containing p' for which Σ is an equivalence-class, then O' and O are coplanar* (that is, if we call an observer-line having Σ as an equivalence-class an "axis" of Σ, then *every pair of axes of Σ is a pair of coplanar world-lines*).

In any event, O and O' can be embedded in a three-dimensional Minkowskian subspace M of our space-time; for if they are not coplanar, then there is a unique three-dimensional affine subspace M containing them (and this is necessarily Minkowskian, since it contains time-like lines), whereas if they are coplanar the two-dimensional subspace that they span can be extended to a three-dimensional subspace M. Any strict automorphism of M can be extended to the entire space-time (e.g., by making it act trivially on an orthogonal complement of M); therefore any such automorphism maps the intersection of Σ and M to itself (this follows obviously from $(S1)$). Consider, first, the family of all rotations of M about O as fixed axis. Under these rotations, p' generates a circle C, whose plane is orthogonal to O, and all of whose points are in Σ (since for any such point p'' we have $S_O(p'', p)$—p itself being fixed under all these rotations). Let l be the tangent-line to C at p'; I claim that O' must be orthogonal to l. Suppose it is not so. Then the plane through p' orthogonal to O' does not contain l. Now, that plane is space-like, and it separates M into two connected components—call them A and B. Since l meets, but does not lie entirely in, this plane, it contains—arbitrarily close to the point p' in which it meets the plane—points of a and points of B. The same must then be true of the circle C, to which l is tangent: in fact, any arc of C which contains p' in its interior and which does not extend as far as the point diametrically opposite to p' is divided by p' into a part that lies in A and one that lies in B; and each of these contains points arbitrarily close to p, and therefore points whose perpendicular distances from O' are arbitrarily small. In particular, *C contains a pair of points, say q in A and q' in B, that are of equal orthogonal distance from O'*. Under rotation of M about O', therefore, q and q' describe circles, in planes orthogonal to O', one in A and one in B, lying on a single cylinder having O' as axis; and all the points one *both these circles, together*, are in Σ, since q and q'—as points of C—lie in Σ, and Σ is invariant under rotations about O'. And this leads to a contradiction of $(S2)$(a): for the generating lines of our cylinder, which are parallel to O' and thus time-like,

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relativity" is quite wonderful; but (a) although he had previously discussed the lack of clarity of the notion of simultaneity for distant events, there is not a single word about distant simultaneity in his great essay [5], *except* for what is implicit in the spatio-temporal transformation equations (represented as changes of coordinates)—and equations, which Poincaré attributes to Lorentz, are indeed those introduced by Lorentz [6]. Further, Poincaré expresses in the introduction to this essay deep reservations about the theory he is presenting, as one that seems artificial and might perhaps some day be simplified by a critical consideration of measurement (so far is he from *offering* such a consideration here!). So (a) there is something to be said for "*Lorentz*-Einstein," rather than "Poincaré-Einstein," but (b) on the other hand, it was Einstein who made a clarification of simultaneity a central theme, and it was Minkowski who geometrized that clarification; hence my preference.

each meet *both* the circle generated by q and that generated by q', whereas these two circles do *not* meet (since one lies in A and the other in B). So it is established that O' is orthogonal to l, as claimed.

But this result means that O' and O are coplanar—that is, it establishes our lemma: for l, as the tangent-line to C, lies in the plane of C—i.e., the plane orthogonal to O; so any plane orthogonal to l is *parallel* to O; but the plane orthogonal to l at the point p', since it contains the center of *C, meets O*, and so must *contain O*; and since this plane, as we have just seen, also contains O', O and O' are indeed coplanar.

There are now two possibilities: O and O' intersect, or they are parallel.[12] Let O'' be any axis of the same equivalence-class Σ that does not lie in the same plane as O and O' (there are such, of course: take an axis at any point of Σ outside that plane—that such points exist is guaranteed, in the light of our assumption that the space-time is at least three-dimensional, by condition $(S2)$(a)). If O and O' are parallel, then since O'' cannot meet both of them it must be parallel to one—but, then, to both; any other axis is non-coplanar with either O and O' or O and O'', and is therefore parallel to O: in other words, in this case *all the axes of* Σ *are parallel*; and by the invariance of S_O under translations in the direction of O, the same must be true of *all* the equivalence-classes of S_O; and then, since there is a strict automorphism that takes O to any other observer-line, for all the equivalence-classes of any observer at all. But this puts us in the situation of Theorem 1 above (either (a) or (b)—the conditions of either are satisfied); so for this case it *does* follow that S_O is Einstein–Minkowski simultaneity: the equivalence-classes of simultaneity are hyperplanes orthogonal to the axes of simultaneity.

Suppose, then, that O and O' intersect. Then O'' (chosen as above) cannot be parallel to either—say, to O—by the immediately preceding result (putting O'' for O' and O' for O''). So we have three non-coplanar lines that intersect in pairs; from which it follows that all three intersect in *one common point* p_0. It follows by an obvious argument[13] (appealing to the fact that if, of three non-coplanar lines, every two intersect, then all three have a point in common) that *all* axes of Σ meet in p_0. (Note that at this point we have established—we have not assumed—that through each point q of Σ there passes *exactly one* axis of Σ: in the previous case, the line parallel to O; in this case, the line p_0q [that p_0 is *distinct* from q, so that there is a determinate line p_0q, is clear, since p_0 is not in Σ].)

To complete the analysis of this case, we must note that any strict automorphism that maps Σ onto itself must also take the set of axes of Σ one-to-one onto itself. For let ϕ be such an automorphism, let p be any point of Σ, let the axis of Σ at p be O, let $\phi(p)$ be p', and let $\phi(O)$ be O'. By the invariance assumption $(S1)$, ϕ transforms S_O to $S_{O'}$; so, since Σ is invariant under ϕ, Σ is an equivalence-class of $S_{O'}$; in other words, O' is an axis of Σ—and therefore, since there is only one axis of Σ at each point of Σ, it is the axis of Σ at p'; and this shows that ϕ does indeed map the set of axes of Σ one-to-one onto itself.

[12] If we adopt the point of view of projective geometry, introducing the "projective completion" of our space-time, these alternatives merge into one.

[13] From the projective point of view, the same argument as in the former case.

But from this it follows that any strict automorphism that maps Σ onto itself *leaves fixed* the point p_0 in which all the axes of Σ meet. Now, if p and p' are any two points of Σ, there is a strict automorphism taking p to p' and taking the axis of Σ at p to that at p' (for, given *any* points p and p', and *any* time-like lines l and l' containing p and p' respectively, there is a strict automorphism of the space-time taking p to p' and l to l'). That automorphism maps Σ to itself, and therefore leaves p_0 fixed. Since a strict automorphism preserves the length and the time-orientation of a time-like vector, the vectors p_0p and p_0p' have the same length and the same time-orientation. Let us call any class of all the time-like vectors that agree in length and time-orientation with a given one v the *temporally oriented radius* $[v]$ determined by v. Then what we have just established is that, *for any point p on an observer-line O, and any point p', if $S_O(p,p')$ holds then the vector p_0p' is time-like and the temporally oriented radius $[p_0p']$ is the same as $[p_0p]$*. We may describe this situation by saying that the equivalence-class Σ of S_O that contains p is contained in the "Minkowski hemisphere" with temporally oriented radius $[p_0p]$; but it must then be the entire hemisphere, since otherwise it will not be the case that every observer-line meets Σ (i.e., assumption $(S2)$(a) will be violated).—However, this formulation is elliptical: it does not identify the point p_0. The answer to that objection is that what must be given (once for all) to determine the function S that assigns to every observer-line its "simultaneity-relation" S_O is *a temporally oriented radius r*. Then, given p, and O containing p, the point p_0 in the foregoing statement—the "center" of the Minkowski hemisphere containing p and having O as an axis—is the unique point *on O* such that $[p_0p] = r$.—It is of course in singling out a particular (time-like) distance and time-orientation that we make essential use of the fact that we have not postulated invariance under change of time-orientation or change of scale.

To sum up, we have established the following:

Theorem 2. *In a Minkowski space-time of three or more dimensions let there be given an assignment, to each observer-line O, of an equivalence relation S_O on the whole space-time, in such a way that:*

($S1$) any strict automorphism that maps O to O' transforms S_O to $S_{O'}$;
($S2$) if Σ is an equivalence-class of S_O then:

(a) *every observer-line O' has one and only one point in common with Σ, and*
(b) *for every point p' of Σ there is an observer-line O' containing p' such that Σ is an equivalence-class of $S_{O'}$ (as well as of S_O);*

—then EITHER (1) *for every O, p, and q, we have that S_O (p,q) holds if and only if the line pq is orthogonal to O,* OR (2) *there is a temporally oriented radius r such that for every O, every p on O, and every point q, we have that S_O (p,q) holds if and only if $[p_0q] = r$, where p_0 is the (unique) point on O for which $[p_0p] = r$.* (Equivalently we may say, in case (2), that *for every O, p, and q, S_O (p,q) holds if and only if, for some p_0 on O, $[p_0p] = [p_0q] = r$.)*

Two ways suggest themselves to strengthen the hypotheses of this theorem so as to eliminate the alternative (2) and leave standard Einstein–Minkowski simultaneity as the only possibility: we may replace $(S2)$(b) either by:

(S2) (b′) *for every pair of space-time points p, p', and every observer-line O containing p, there is an observer-line O' containing p' such that $S_{O'}$ coincides with S_O;*

or by*:*

(S2) (b″) *for every observer-line O, every equivalence-class Σ of S_O and every space-time point p, there is an observer-line O' containing p such that Σ is an equivalence-class of $S_{O'}$ (as well as of S_O).*

—The first of these alternative assumptions obviously rules out the alternative conclusion (2), since in the latter the *families* of equivalence-classes of any two distinct observer-lines have, *ipso facto*, their "moving centers" on different lines (the centers coincide only where the lines intersect). The second rules out conclusion (2) because for Σ to be an equivalence-class of $S_{O'}$ it is necessary that O' pass through the center p_0 of the "Minkowski hemisphere" Σ,[14] and of course not every observer-line does so.

Let us now consider a two-dimensional Minkowski space-time. This case differs, as has been intimated, in important ways from that of any higher dimension, and I think the difference is worth noting. (One point is obvious from the outset: namely, that the lemma we have exploited in the proof of Theorem 2 holds trivially in two dimensions—all the axes of a simultaneity-class are coplanar, because *everything* is coplanar; but this fact, in just this case, is of no help at all, and a quite different line of attack is required.) I shall begin by reviewing a few basic facts about the geometry of a Minkowski space-time of two dimensions:

(1) In two dimensions, there is complete symmetry in the geometry as between space and time, since the fundamental quadratic form has, in diagonal form, one positive and one negative coefficient.[15] It would, indeed, be possible in this case to extend the notion of an "automorphism" of the space so as to include an interchange of "space" and "time."[16] One aspect of this (as it were)

[14] This condition is also sufficient.

[15] There is in the literature some variation in the choice of signs: in the older convention, introduced by Minkowski (the "time-coordinate" as an imaginary number), the negative sign is assigned to the one temporal dimension, the positive sign to the n (for physics, 3) spatial ones; but the reverse choice is often made. This *is*, in the clearest sense, a pure matter of "convention": which of them one adopts makes no real difference. Accordingly, in the above discussion, I have never indicated a preference on this point: I have referred to vectors and subspaces as "time-like" or "space-like," without assigning an algebraic sign to the one or the other. I shall continue to do this in what follows.

[16] As we know, the wider sense of "automorphism" usual in mathematics for Minkowski space-times includes change of scale; this is tantamount to regarding, as characterizing the geometry, not a given non-degenerate quadratic form of appropriate signature, but a *class* of such forms, arising from one another through multiplication by arbitrary positive real factors. Nothing prevents one from admitting instead multiplication by arbitrary *nonzero* real factors. This, in the general case, of $n+1$ dimensions with $n > 1$, would make no difference at all to the theory; but in the case $n = 1$, it would automatically allow as automorphisms maps that preserve the linear (more exactly, affine) structure, preserve the relation of orthogonality, but take "time-like" vectors to "space-like" ones and *vice versa*.

"time-likeness of space" in a two-dimensional space-time is that, just as it is possible to divide time-like vectors into two classes that are topologically separated from each other, so this is also possible for space-like vectors. In fact, under "proper automorphisms," the non-zero vectors of a two-dimensional Minkowski space-time fall into eight distinct classes: two of time-like vectors (we may call them "future-pointing" and "past-pointing"), two of space-like vectors ("right-pointing" and "left-pointing"), and four classes of null vectors ("right-and-future pointing," "left-and-future pointing," "left-and-past pointing," "right-and-past pointing"): one non-zero vector can be taken to another by a proper automorphism if and only if the two belong to the same class.[17] I shall call two non-zero vectors—or two lines—"like," or "of the same character," if they are both time-like, both space-like, or both null; and shall call (as above) two like vectors "of the same class" if they are similarly oriented—i.e., if they are "equivalent" under proper automorphisms.—Two vectors are like if and only if their inner products with themselves have like signs (here zero is to be counted as a sign in its own right, distinct from plus and minus); two non-null like vectors are of the same class if and only if their inner product has the same sign as the inner product of each with itself.[18]—The fact that proper automorphisms—and, *a fortiori*, strict automorphisms—preserve spatial as well as temporal orientation *of individual vectors* will prove to be important in the following.

(2) We shall have occasion to make use of the following facts about triangles—equivalently, about three vectors of which one is the sum of the other two—in a Minkowski plane:

(a) Let two vectorial sides—*AB, BC*—of a triangle be space-like and of the same class; then the side *AC* (their vector-sum) is also space-like and of the same class with them.
—This is perhaps obvious; to prove it, we have—representing the inner product by angle brackets—that, since $AC = AB + BC$, $< AC, AC >= < AB, AB > +2 < AB, BC > + < BC, BC >$. By the criterion stated in (1) above, all three terms have the same sign; therefore the inner product of *AC* with itself has the same sign as those of *AB* and of *BC* with themselves: *AC* is space-like. And $< AC, AB >=< AB, AB > + < BC, AB >$—again a sum of terms of like sign—so it, too, has the same sign as the inner product of *AB* with itself, which shows that *AC* and *AB* are of the same class.

(b) If *A, B, C* are non-collinear points and *D* is a point on the space-like line *BC* such that *AD* is orthogonal to *BC*, then:

[17] To avoid a possible misunderstanding: "automorphism" is *not* here used in the extended sense mentioned in the previous note; and indeed, even if it were, since "proper" automorphisms are those that belong to the connected component of the identity in the Lie group of all automorphisms, the ones "interchanging time and space" would be excluded.

[18] Although it will be of no importance in what follows, it perhaps ought to be noted explicitly that this criterion fails for null vectors: two null-vectors directed along the same line have inner product zero whether their "senses" are the same or opposite.

(1) the vectors AB and AC are of like character and equal in length if and only if D is the midpoint of the segment BC (i.e., if and only if A is on the perpendicular bisector of BC);

(2) if this is indeed the case—i.e., if D is the midpoint of BC—then:

(i) if AB and AC are space-like, then BA, AC, and BC are all of the same class, and the lengths of BA and of AC are *less than half* that of BC;

(ii) if AB and AC are time-like, then AB, AD, and AC are all of the same class.

—For, setting for convenience $x = AD,\ y = BD,\ z = DC$ (all as vectors), and noting that $AB = x - y$ and $AC = x + z$, the necessary and sufficient condition for AB and AC to be of like character and equal in length is that $< x-y,\ x-y >=< x+z,\ x+z >$; i.e., in view of the orthogonality of x to y and to z, that $< x,\ x > + < y,\ y >=< x,\ x > + < z,\ z >$; i.e., that $< y,\ y >=< z,\ z >$; and since y and z are in the same line, this means that $y = \pm z$. But $y = -z$ is impossible, because that would mean $BD + DC = 0$, i.e., $C = B$, whereas our assumption that A, B, and C are non-collinear implies that these are three distinct points. So the condition for AB and AC to be of like character reduces to $y = z$, i.e., $BD = DC$, which is to say: D is the midpoint of BC; so (1) is proved.—Proceeding to (2)(i), we are now to assume that $z = y$ and, further, that AB and AC are space-like, which is to say that $< x-y,\ x-y >$ and $< x+y,\ x+y >$ (which of course are both equal to $< x,\ x > + < y,\ y >$) have the same sign as $< y,\ y >$ (this in view of the fact that y is space-like). What has to be proved is that $y - x,\ x + z$ (i.e., $x + y$), and $y + z$ (i.e., $2y$) are of the same class. It suffices to show that each of the first two is of the same class with the third; which is to say, that the inner products $< y-x,\ 2y >$ and $< y+x,\ 2y >$ have each the same sign as that of the inner product of a space-like vector with itself. But—again, since x and y are orthogonal—these are both equal to $2 < y,\ y >$ and, y being space-like, the point is established. The claim about the lengths of BA and of AC—that is, of $y - x$ and of $x + y$—follows almost immediately from the expression for both of them, $< y,\ y > + < x,\ x >$: since the terms of this sum are of opposite sign and the sum, by hypothesis, has the sign of $< y,\ y >$, it is smaller in absolute value than $< y,\ y >$; so the (common) length of $y - x$ and of $y + x$ is less than that of y—i.e., than half the length of BC.—Finally, as for (2)(ii), we now have to suppose that $< x,\ x > + < y,\ y >$ has the same sign as $< x,\ x >$, and to show that $x - y$, x, and $x + y$ are of the same class. Analogously to the case of (i), it suffices to show that the inner product with x of each of the other two has the same sign as that of x with itself; but this is even more obvious than in the other case: each of these inner products is *equal* to $< x,\ x >$. The proof of (b) is complete.

(c) In an isosceles triangle ABC with the two vectorial sides AB, AC space-like, equal in length, and of the same class, the "base" BC is time-like.

—This is most easily seen by considering the sum $AB + AC$, which by (a) is space-like. Its inner product with BC—since the latter is $AC - AB$ [for $AB +$

$BC = AC$]—is $< AB, AC > - < AB, AB > + < AC, AC > - < AC, AB >$. The first and last terms of this sum cancel, by the commutativity of the inner product, and so do the middle two terms, since AB and AC are like and of equal length; so the whole is zero—i.e., BC is orthogonal to the space-like vector $AB + AC$.

(3) Finally, some facts about the strict automorphisms of a Minkowski plane (or the connected component of the identity in its Poincaré group):

(a) There are two kinds of automorphisms besides the identity: the translations, which have no fixed points and which leave invariant one complete family of parallel lines; and the "Minkowski rotations" ("Lorentz transformations," "boosts"), each of which has a unique fixed-point and leaves invariant only the two null-lines through the fixed-point. If p and p' are any two points, there is a unique translation that maps p to p'. If p and p' are any two points, and if l and l' are any two non-null lines of the same character through p and p' respectively, there is a unique strict automorphism that takes p to p' and l to l'.[19] For there is a translation that takes p to p', and this may be followed by a "boost" that leaves p' fixed and takes the image of l under the translation to l'. (That the resulting automorphism is unique follows from the fact that the identity is the only automorphism that leaves a point fixed and a non-null line through that point invariant; this—which is not hard to prove—I here simply take for granted.)

(b) If one considers all the strict automorphisms having a given fixed-point (here including the identity), they constitute a one-parameter subgroup of the Poincaré group. Unlike the case of Euclidean rotations, a one-parameter group of "Minkowski rotations" is non-compact—as the parameter varies from $-\infty$ to $+\infty$, the mapping from parameter values to group elements is one-to-one. In spite of this fact, one can—as in the Euclidean case—make a "natural" choice of parameter, which in the Euclidean case is the *angle* of rotation (the "natural" choice—there is not a unique one!—may be the radian-measure, or the measure by fractions of a full rotation).[20] In any event, the correspondence of the parameters to the rotations they parametrize is of such a kind that multiplying the parameter by, say, a positive integer n corresponds to "composing a rotation with itself" $n - 1$ times (the "minus one" comes from the fact that, e.g., "composing an operation with itself" *once* means "performing that operation" *twice*). Multiplication

[19] Although, once again, it is of no importance for us, let it be remarked that this is not the case if L and L' are both null: in this case, these lines must also be, let me say, "similarly inclined" (either they must both go from past and left to future and right, or both from past and right to future and left, for there to be any automorphism that takes one to the other; and then there will be infinitely many automorphisms that take p to p' and L to L').

[20] It is striking that in spite of the fact that in the Minkowski case there is no such thing as a "full rotation," there is nevertheless a "natural" analogue to the angle; but this is just one more manifestation of the marvelous interconnection of the trigonometric functions and the exponential function. (I trust the reader will forgive this gratuitous advertisement of the splendors of [even the rather elementary part of] mathematics.)

of the parameter by -1 amounts to taking the *inverse* of the given rotation (from which follows the interpretation of multiplying by any negative integer). Analogously—but this actually gives us something new—multiplying the parameter corresponding to a given rotation ϕ by the *reciprocal* of an integer n, we obtain a rotation which, "performed n times," results in the original rotation ϕ. Since the nth iterate is indicated by n as an exponent, the procedure just described yields what one might call the "nth root" of ϕ; but I shall call this instead—bearing in mind the parameter (the "quasi-angle") of the rotation—the result of *dividing* the rotation by n; and by an obvious extension, we obtain the interpretation of multiplication by any rational number.—Multiplication by *irrational* real numbers is an *entirely* new generalization of the simple idea of "composition of an operator with itself." (As will be seen below, what we shall have to consider is primarily the case $n = 2$ or a power of 2, and then multiplication by any rational number whose denominator is a power of 2; and I shall speak of "halving" the rotation, or of its "successive halving," etc., rather than of "taking square roots," or "taking to a rational power," etc.)

(c) The analogous situation for a translation is simpler: a translation (different from the identity) is represented by a (non-zero) vector, and when translations are considered separately, their composition is represented as *addition* of vectors (but when they are considered together with the other elements of the Poincaré group their composition with any group-element—translations themselves included—is represented multiplicatively). So, treating a translation and its compositions with *itself* (and iterations thereof), these amount to multiplications by a positive integer; the extension to positive and negative rational (or, for that matter, irrational) numbers remains in the domain familiar from the ordinary treatment of vectors, and the structure of the one-parameter group "generated by" a given non-zero translation is obvious. (Let it be noted that every non-zero multiple of a Minkowski rotation is a rotation and every non-zero multiple of a translation is a translation.)

In treating the higher-dimensional case, we excluded from the start (via condition $(S2)$(a)) the possibility that a simultaneity equivalence-class contains two distinct points with time-like separation, but we did not exclude in advance the possibility that such a class contains two distinct points with null separation—this emerged as a *consequence* of our hypotheses. In the two-dimensional case, if we wish to avoid this possibility in the end, we have to strengthen the assumptions (this will emerge clearly when the analysis is complete). Therefore, for the following discussion, I wish to strengthen condition $(S2)$, replacing clause (a) by the following: (a$'$)(i) *every observer-line O' meets Σ*; (ii) *any two distinct points of Σ have space-like separation*. The remaining hypotheses of Theorem 2 remain unaltered—except, of course, that we are now to consider a Minkowski space-time of two dimensions.

Let Σ again be an equivalence-class, containing the point p, of the "simultaneity" relation S_O, where p belongs to the axis O. It will be useful in the present case to begin by proving that there is no other axis of Σ at p. Suppose there were an axis O', distinct from O, containing p. Let q be a point of Σ with space-like separation from p (that there are such points follows immediately from condition $(S2)(\mathrm{a}')(\mathrm{i})$ and (ii), and from the fact that there are observer-lines that do not pass through p). Consider, now, the Minkowski rotation about p that takes O to O'. The image q' of q under this rotation belongs to Σ: for q, as a point of Σ, satisfies $S_O(p,q)$; therefore, by the invariance condition $(S1)$, q' satisfies $S_{O'}(p,q)$—and since O' is (assumed to be) an axis of Σ, q' belongs to Σ. On the other hand, the rotation about p that took q to q' took the vector pq to pq'; and since a rotation preserves the character, the length, and the class of a vector, pq' is space-like, equal in length to pq, and of the same character as pq. Therefore, by (2)(c) in the foregoing discussion, *if* there is at p an axis O' distinct from O—which implies that q' is distinct from q—there is a time-like line qq' containing two distinct points of Σ; and since this violates condition $(S2)(\mathrm{a}')$(ii), there cannot be an axis at p distinct from O.

Next we shall see that any strict automorphism that maps Σ *into* itself must (a) take the set of *axes* of Σ into itself, and must (b) take both the set of points, and the set of axes, of Σ, one-to-one *onto* themselves. For let ϕ be such an automorphism, let p be a point of Σ, let O be the axis of Σ at p, let p' be $\phi(p)$ and let O' be the image $\phi(O)$ of O under ϕ. We must first show that O' is an axis of Σ; so let q be any point such that $S_{O'}(p',q)$ holds. Since ϕ^{-1}, the inverse of ϕ, is (of course) also a strict automorphism, we have by invariance that $S_O(p,\phi^{-1}(q))$ holds; therefore $\phi^{-1}(q)$ belongs to Σ, so since ϕ maps Σ into itself $\phi(\phi^{-1}(q))$—which is to say, q—belongs to Σ; and this means that O' is indeed an axis of Σ. Now let q be any point of Σ, and let ϕ, p, O, and O' be as in the discussion of clause (a). We have just seen that O' is an axis of Σ, so $S_{O'}(p',q)$ holds (as was *assumed* of q in that discussion); therefore it follows again (or still!) that $\phi^{-1}(q)$ belongs to Σ, and accordingly that q is the image under ϕ of a point of Σ, which shows that ϕ maps Σ *onto* itself. Further, ϕ (as we already know) takes axes of Σ to axes of Σ; therefore it takes the axis of Σ at $\phi^{-1}(q)$ to that at q; and since q was an arbitrary point of Σ—and so the axis of Σ at q is an arbitrary axis of Σ—every axis of Σ is the image under ϕ of some axis of Σ, so the mapping by ϕ of the set of all such axes is *onto* that same set. Finally, as an automorphism of the space-time, ϕ is automatically one-to-one on *any* set of space-time points; and although an automorphism is *not* necessarily one-to-one on the set of time-like lines ("observer-lines"), an automorphism of Σ is one-to-one on its axes because the axes correspond one-to-one with their points of intersection with Σ, and the automorphism is one-to-one on that set of points. Thus clause (b) too has been fully demonstrated.

With all these perhaps tedious but comparatively trivial points now established, the main conclusion of the present an analysis could be rather quickly reached, on the basis of the discussion of one-parameter groups under (3)(b) and (c) above. But since just that part of the preliminary discussion contented itself with a vague indication of the proofs of its claims—by means, namely, in (3)(b), of the notion of "quasi-angles" (which is to say, the "hyperbolic trigonometry" of the Minkowskian plane), it seems preferable to base the results instead on more elementary geometric constructions.

Let, then, Σ still be an equivalence-class of a "simultaneity-relation," let p and p' be points of Σ, let O and O' be its axes at p and p' respectively, and let ϕ be the strict automorphism that takes p to p' and O to O'. Then ϕ maps Σ into itself (and therefore, by the preceding result, onto itself), because if q is in Σ if $q' = \phi(q)$, we have $S_O(p,q)$; therefore $S_{O'}(p',q')$; therefore q' is in Σ. We mean now to "halve" ϕ. To this end, consider the perpendicular bisector of the space-like line-segment pp'. This is a time-like line (cf. n. 21 above), and therefore contains a (unique) point p'' of Σ; let the axis of Σ at p'' be O'', and let the strict automorphism that takes p to p'' and O to O'' be ψ.[21] I maintain that ψ "*halves*" ϕ.—Proof: First, if $q = \psi p''$) then q is a point of Σ, and $p''q$, as the image under ψ of pp'', is of the same character and length as the latter; but therefore, by (2)(b) of the preliminary discussion, of the same character and length as $p''p'$ as well. From this in turn, by (2)(c),[22] it follows that unless $q = p'$, qp' will be a time-like line containing two distinct points of Σ. Since this last is impossible, we must have $q = p'$; but then also the image under ψ of O'' must be O'; so since ψ takes p to p'' and p'' to p', and takes O to O'' and O'' to O', its "double" ψ^2 takes p to p' and O to O'; but these are the defining characteristics of ϕ, so the point is established.

Now observe that if we allow ϕ to "act iteratively on p," a countable set of points will be generated—all on Σ, equally spaced—proceeding from p on the one side; that ϕ^{-1} will similarly generate such a set proceeding from p on the opposite side; and that allowing ψ and its inverse to act similarly, we shall obtain another such pair of sets, which include all the points of the first pair, with new points interpolated between every two adjacent points of the first sets. The (equal) distances between successive points of the "refined" system will be *no more than half* that between successive points of the first system: namely, if three successive points such as $p,\ p'',\ p'$ above form a triangle, the distances $|pp''|$ and $|p''p'|$ are *less than half* $|pp'|$ (see (2)(b) of the preliminary discussion), whereas if these points are on a single straight line then p'' is the midpoint of pp', so the former distances are *exactly* half the latter one.

We next conceive this process of "bisection" of the automorphism and the generation of new points on Σ to be iterated without bound. The result is a system of points "densely" distributed on Σ, in the sense that any one point has, on each "spatial side" of itself, others whose (space-like) distances from it are arbitrarily small.[23] We must now consider the geometric nature of the locus of these points.

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[21] Note, by the way, that in light of what we now know we could characterize ψ equivalently as the strict automorphism that takes p to p'' *and maps* Σ *into* (or: *onto*) *itself*. By the same token, ϕ can be characterized as the strict automorphism that takes p to p' and maps Σ into (or: onto) itself.

[22] Here the reader may (should!) have a sense of *déjà vu*.

[23] Caution!—We cannot immediately conclude that these points are "dense *in the topology induced on* Σ *by that of the Minkowski plane*. For the space-like distance is not a metric in the standard sense of the theory of metric spaces—this because the triangle inequality is, as we have seen, "*the wrong one*." Indeed, it is quite possible for a sequence of points to be (as it were) a "Cauchy sequence in the space-like distance," but not to converge; or for the distances of those points from a given point p to converge to zero, but the points *not* to converge to p. (This last is especially easy to see: let the "Cauchy sequence" converge to a point q, distinct from p, with *null separation* from p; then the "distances" from p will go to zero, but the sequence will not converge to p.) We shall have to (and we shall be able to) circumvent this difficulty eventually.

With Σ, p, p', p'', O, O', O'', ϕ, and ψ as above, suppose first that p, p'', and p' are collinear—so that p'' is the midpoint of pp'. Then the vectors pp'' and $p''p'$ are *the same*; so ϕ, mapping p to p'' and p'' to p', and therefore taking the vector pp'' to $p''p'$, leaves that vector *fixed*.[24] But a strict automorphism that leaves a non-null vector fixed also (a) leaves fixed all its multiples by scalars, and (b) leaves fixed any vector orthogonal to the given one (since *strict* automorphisms preserve the relation of orthogonality and preserve the length, character, *and class* of any vector).[25] But given a non-null vector, every vector can be represented as a sum of a multiple of that vector and a vector orthogonal to it; and automorphisms preserve sums; so we conclude that a strict automorphism that leaves a non-nulll vector fixed leaves every vector fixed. However, it is easy to see that an automorphism that leaves every vector fixed is a translation. So—in the case in which p, p', and p'' are collinear—the automorphism ψ is a translation—and, in consequence, all its powers, and also its inverse and all the powers of its inverse are translations.

Before we draw the (almost obvious) conclusion from this about the geometric character of the set Σ, it will prove best to consider the other case—that, namely, in which the points in question are *not* collinear, so that the segments—and the vectors (we have been using the same notation for both)—pp', pp'', and $p''p'$ are distinct, and the segments are the (space-like) sides of a (*n* isosceles) triangle. Consider, then, the perpendicular bisectors of the equal sides pp'' and $p''p'$. These meet (since the lines pp'' and $p''p'$ are not parallel, lines orthogonal to them are also not parallel) in a point p_0 that is equidistant from the end-points of both—i.e., from p, p'', and p'—and that therefore lies on the perpendicular bisector of pp' (see (2)(b) of the preliminary discussion). Since that perpendicular bisector is time-like, the vector p_0p'' is time-like; I claim that the vectors p_0p and p_0p' must then also be time-like, and of the same class as p_0p''. Indeed, we know already, by (2)(b) of the preliminary discussion applied to the triangle whose vertices are p_0, p, and p'', that p_0p and p_0p'' are of the same character, since p_0 lies on the perpendicular bisector of the space-like pp''; so p_0p must be time-like, because p_0p'' is so; and analogously for p_0p'. But then we know —again by (2)(b), now applied to the triangle whose vertices are p_0, p, *and* p', and in which p'' is the midpoint of pp'—that p_0p, p_0p', and p_0p'' are indeed of the same class.

Now, the strict automorphism ψ takes p to p'' and p'' to p'. It therefore takes the perpendicular bisector of pp'' to that of $p''p'$. Let r be the temporally oriented

[24] Lest there be confusion: an automorphism of a Minkowski space is to be regarded as acting *both* on the space of points *and* on the associated vector space (thus, in a more precise sense, as a *pair* of maps [which we nonetheless designate by the same symbol])—the connection between the two actions (or the two maps) being that if the *points* A and B are taken to A' and B' respectively, then the *vector* AB is taken to $A'B'$.

[25] Two points of clarification: (1) strictness is required only for the preservation of "class": without that, vectors orthogonal to a fixed one need not themselves be fixed, they may be "reversed"—i.e., multiplied by -1; (2) "non-null" is important here because a vector orthogonal to a null-vector in a Minkowski plane is a multiple of that null-vector: it is not true that every vector is a linear combination of the first vector and the second. (It *does* remain true for a non-zero null-vector—in two dimensions—that a strict automorphism that leaves one fixed leaves all vectors fixed; but another argument would be required to prove this, and we have no need of the fact.)

radius $[p_0 p]$; then the point p_0 may be characterized as the unique point q on the perpendicular bisector of pp'' for which $[qp] = r$; it may also be characterized as the unique point q on that perpendicular bisector for which $[qp''] = r$. And by the same token, p_0 may be characterized as the unique point q' on the perpendicular bisector of $p''p'$ for which $[q'p''] = r$. But ψ takes any point q on the perpendicular bisector of pp'' for which $[qp] = r$ to a point q' on the perpendicular bisector of $p''p'$ for which $[q'p''] = r$; that is, ψ takes p_0 to p_0 : p_0 *is a fixed-point of* ψ (and, as we know, there cannot be more than one such: p_0 is *the* fixed-point of ψ (which, incidentally, by this very argument *possesses* a fixed point: i.e., ψ *is a Minkowski rotation*.

This conclusion leads to a far-reaching consequence for our *other* case—that in which $p,\ p'$, and p'' are collinear. We saw that in that case ψ is a translation. We can now infer that the result of *halving* ψ is *again* a translation. For we saw, in our previous analysis, that if, starting from $p,\ p'$, and the strict automorphism ϕ that takes p to p' and maps Σ into itself (cf. n. 22 above), we construct the point p'' and the automorphism ψ that halves ϕ, then if $p,\ p',\ p''$ are collinear, ψ is a translation, and has no fixed-point; we have now seen that if those points are *not* collinear, ψ is a rotation, and *does* have a fixed-point. Applying this to the halving of ψ (with p'' playing the role that p' did previously, and with a new "third point" for p''), we see that unless the new triad of points is collinear, the result of halving ψ will be a rotation—a transformation that has a fixed-point. But this is impossible; for if a transformation has a fixed-point, so does the result of "doubling" it. Therefore, as claimed, the result of halving ψ will in that first case be again a translation; *and so on ad infinitum*. And it follows immediately that not only the "backwards and forwards *sequences* of points" that we constructed from ϕ and refined using ψ, but all the points of the subsequently constructed "dense" system in Σ *lie on a single straight line*.

The conclusion for the second case—in which the first three points of the construction, $p,\ p',\ p$ are *non*-collinear—is obviously analogous: If ψ has a fixed-point p_0, all the subsequent results of halving must likewise have, not only a fixed-point, but *the same* fixed-point p_0—the result of doubling a rotation has the same fixed-point—or "center"—as the original rotation, and therefore the result of halving a rotation must have the same fixed point as the original rotation. It follows that the points of the "dense" array in Σ in this case *all lie on a "Minkowski semicircle" having the temporally oriented radius* $[p_0 p]$.

Two matters remain to be treated. First, we are obviously led to think that Σ itself just *is* the straight line or "semicircle" concerned; but this has to be proved. And then, we have so far determined nothing about the *axes* of Σ: in the higher-dimensional case, the fact that these axes (in the "hemispherical" case) have all a single point of intersection was crucial to our whole argument; but in the two-dimensional case, we have made very limited use of arguments involving the axes, and have so far draw *no conclusion whatever* about their geometrical configuration.

In order to deal with the first problem, there is one obstruction we have to remove. We have indeed seen that the range of points that lie on, in the one case a straight line, in the other a circle, extends "infinitely" in both directions, and has “everywhere” points as close together as one wants; but “infinitely” and “everywhere” here are in an important respect misleading: we have shown that there are *infinitely many* points, in each direction, in our array; we have *not* shown—now taking advantage of the possibility of using the geometric loci, the straight line or the “semicircle,” as a sort of standard—that our range of points extends from the initial point p, in both directions, *past any given point of the line or “semicircle” that we care to name*.

In the case of the straight line, this is trivial to deal with: at the very first stage, when we iterate the translation ϕ or ψ, we *do* obtain points arbitrarily far from p along the line in either direction; and of course this continues to be true at every stage of the subdivision. In the case of the “semicircle,” the situation is quite analogous: the iteration of a rotation leads to points, in both directions from p along the curve, that extend (in point of the natural ordering of points on an “open” curve) *past* any given point. To prove this is not hard, but going into the details of a proof would not afford any new insight into the matter—any reader who is unfamiliar with the fact stated and who cares to have a proof should be able to find one—so I shall now take this for granted. More precisely, I shall make use of the fact that if one is given any positive real number a, and any point q of the straight line or “semicircle,” we can find a sequence of points of our array, starting from p and continuing past q, such that the space-like separation between any two successive points of the sequence is less than a. Then, from this sequence, one can select *two* points, say q' and q'', *one on each side of* q, and with space-like separation less than a.

If this *is* granted, the question of the full geometric locus can be settled at once. Let q be any point on our line or “semicircle” that contains our “generated” array (q is not assumed to belong to the array); it is to be shown that q belongs to Σ. To this end, let l be any time-like line through q. By condition $(S2)(\mathrm{a}')(\mathrm{i})$, l contains a point of Σ. By the fact stated just above, there are points q', q'' of our constructed point-array—points belonging both to Σ and to the line or “semicircle”—*on both sides of* the point of l in Σ and “arbitrarily close” to one another. Both q' and q'' have space-like separation from the point of l in Σ (since *all* points of Σ have space-like separation, by $(2)(\mathrm{a}')(\mathrm{ii})$). But it is obvious that a point x that has space-like separation from points on the line or “semicircle” lying arbitrarily close to one another and on both spatial sides of x can only be a point that itself lies on the line or “semicircle”. Therefore the point of l in Σ must be the point q in which l meets that geometric locus—as was to be shown.

Now that we know that every point of our line or “semicircle” belongs to Σ, we can easily show that these are the *only* points of Σ—i.e., that Σ *is* the line or “semicircle”: for every point q of Σ belongs to some time-like world-line; and every time-like world-line meets the line or “semicircle” in some point q'; that point of intersection q' belongs to Σ (as we have just seen); but l has only *one* point in common with Σ. So the point q of Σ through which L passes is the same as the point q' in which l meets the line or “semicircle”: the analysis of the geometrical nature of Σ is complete.

As to the axes of Σ—and this is the only point on which we shall find a difference *in the end result* from the higher dimensions—one can choose the axis at a given point q of Σ *arbitrarily*; that is, it can be any time-like line l through p. For the choice of such a line as axis can be specified in the following "objectively geometrical" way: Choose—"in advance" and once for all—a spatially oriented radius s, of absolute length less than unity. (It is obvious how "spatially oriented radius" is to be defined, except for the new proviso that here we allow the vector 0 to count as "space-like" and "oriented"; it therefore constitutes a class by itself—the spatially oriented radius 0. By the "absolute length" of a spatially or temporally oriented radius r we of course mean the absolute value of the length of any vector in the class r). Then let u be the time-like, "future-pointing," unit vector normal to the surface Σ at the point p; let v be the unique vector orthogonal to u such that $[v] = s$ (of course, if $s = 0$, v is the zero vector); and let l be the line through p in the direction of the vector $u + v$ (any time-like line through p—and only such lines—can be described in this way). Now, it can be seen without great difficulty (I forgo details here) that if we start with one particular locus Σ of the kind already determined; if we then, having chosen the spatially oriented radius s once for all, assign to every point p of Σ the corresponding line l as "axis of simultaneity" for Σ at p; and if, finally, we apply to this configuration all possible strict automorphisms of the two-dimensional Minkowski space-time; then the resulting configuration determines an assignment to every time-like line l of a relation S_l satisfying all our conditions.[26]

If one equivalence-class of our system is a straight (necessarily space-like) world-line, then they all are; all those that belong to some one axis are parallel; and all the axes of any one equivalence-class are parallel. The axes of different systems of equivalence-classes—that is, the observer-lines O with different associated relations S_O—are "inclined at the same angle" to the lines normal to their equivalence-classes: this can be taken to mean simply that there is a strict automorphism taking the one normal line to the other, and at the same time taking the one axis to the other.

If one of the equivalence-classes is a "Minkowski semicircle," then they all are—and, moreover, they all have the same temporally oriented radius r. No two equivalence-classes have the same set of axes: as in the higher-dimensional case, each observer-line has its own family of equivalence-classes. And just as in the case of the straight-line equivalence-classes, each axis of an equivalence-class is "inclined at a given angle"—the same for all axes and for all equivalence-classes—to the line normal to the equivalence-class at its point of intersection with the axis.

It is worth noting that whereas the possibility of "Minkowski-semicircular" equivalence-classes is tied to the fact that we are not requiring scale-invariance—so that we are free to choose a temporally oriented radius $r \neq 0$—the possibility, in the

[26] More precisely, for any observer-line l and any point p, there will be one and only one set Σ' that is an image of (the original) Σ under a strict automorphism such that p belongs to Σ', l is an axis of Σ' at its point of intersection with Σ', and for any points q, $S_l(p,q)$ holds if and only if q belongs to Σ'.—There are of course many details to check to justify the statement that this S satisfies all our conditions; the only matter that may seem doubtful is whether, for any given time-like line l and associated equivalence-class Σ with l oblique to the normal to Σ' at their point of intersection, the system of all translates of Σ' in the direction of l constitutes a foliation of the space—i.e., whether every point belongs to one and only one such translate. This can be made transparently so, in the usual Euclidean picture of the Minkowskian plane, through a judicious transformation of coordinates.

case of straight-line equivalence-classes, of "inclined axes," does *not* depend upon that weakening of the invariance requirement (although it *does* depend on giving up invariance under spatial or temporal reflections); for the construction of the axis described above may be modified as follows: instead of taking for u the *unit* future-pointing normal vector, take for it *any* future-pointing normal vector; and then take (as a preliminary step) a vector u' orthogonal to u, having as inner product with itself the negative of the inner product of u with itself and pointing "right," and then, having chosen (once for all) a non-zero (signed) real number s, let $v = su'$.

Either of the strengthened conditions $(S2)(\mathrm{b}')$ or $(S2)(\mathrm{b}'')$ will again restrict the equivalence-classes to the "straight" ones only—i.e., to the same classes given by the Einstein–Minkowski relative simultaneity relations. *But the relations of simultaneity relative to an observer are* ***not*** *necessarily those of Einstein–Minkowski*: for the *axes* retain their degree of arbitrariness: they can be "inclined at any given angle" (given once for all, that is) to the lines normal to the equivalence-classes.

One further fact seems worth pointing out, regarding the contrast with the higher-dimensional case: There, when the simultaneity set was a "hemisphere," all its axes of simultaneity (which of course were just the diameters of the corresponding "sphere") had a common point of intersection: the center of the "sphere." But in two dimensions, for an s different from zero, the axes of simultaneity of a "semicircle" do *not* all meet in a single point.—What particular geometrical configuration the set of axes form for r different from zero is a question that may here be left for the entertainment of Platonic philosophers.[27]

Summing up, our results for the case of a two-dimensional Minkowski space-time are as follows:

Theorem 3. *In a two-dimensional Minkowski space let there be given an assignment, to each observer-line O, of an equivalence relation S_O on the whole space-time, in such a way that:*

($S1$) any strict automorphism that maps O to O' transforms S_O to $S_{O'}$;
($S2$) if Σ is an equivalence-class of S_O then:

(a$'$) (i) *every observer-line O' meets Σ;*
(ii) *any two distinct points of Σ have space-like separation, and*
(b) *for every point p' of Σ there is an observer-line O' containing p' such that Σ is an equivalence-class of $S_{O'}$ (as well as of S_O);*

—then (1) *the system of equivalence-classes for one observer-line is either* (a) *a family of parallel space-like straight lines, or* (b) *a family of "Minkowski semicircles" of given temporally oriented radius r;* (2) *these alternatives hold "uniformly" for all observer-lines—that is, either* (a) *holds for all, or* (b) *holds for all—and then with the same time-oriented radius r for all the observers; and* (3) *in either case, there is a fixed spatially oriented radius s of absolute length less than unity (it may be zero), such that for any point p of any equivalence-class Σ, if u is the unit normal vector to Σ at p, and if v is a spatially oriented vector orthogonal to u such that the*

[27] That is, lovers of geometry. (I have stated the result in Supplementary Note 4.)

oriented radius [v] is s, the axis of Σ *at p is the line through p in the direction of the vector* $u+v$*.—If condition* $(S2)$(b) *is replaced by either* $(S2)(\mathrm{b}')$ *or* $(S2)(\mathrm{b}'')$—(for which see the discussion following Theorem 2 above*)—then the alternative* (b) *is excluded: all the equivalence-classes are straight lines.*

Addendum*: In the case of straight-line equivalence classes, invariance holds under the wider class of all "proper" automorphisms; for this wider class, modify clause* (3) *as follows:* $(3')$ *there is a fixed non-zero (signed) real number s such that for any point p of any equivalence-class* Σ*, if u is any future-pointing normal vector and* u' *is a right-pointing vector orthogonal to u, the absolute value of whose inner product with itself is equal to the absolute vale of the inner product of u with itself, the axis of* Σ *at p is the line through p in the direction of the vector* $u+v$*.*[28]

2 Remarks on the Controversy

It is certainly no new observation that philosophical controversy is often vitiated by the fact that the disputants argue at cross-purposes; in particular, that they use the same words with different meanings. Locke was far from the first, Wittgenstein, Carnap, and Quine far from the last to see in a lack of clarity in linguistic use a prime source of the apparent intransigence of philosophical problems.[29] Nor, considering that the pointing out of this (in principle after all fairly obvious) fact has not so far notably lessened the evil, is it at all likely—to compare a minor writer with major ones—that I shall be the last one to do so either. Nonetheless, as I have indicated in the opening paragraph, I have *some* hope of helping a *little* to clarify this particular issue.

The most crucial notion that cries out here for clarification is that of the "conventional." Poincaré, whose emphasis upon this notion is the beginning of its latter-day philosophical prominence, argued that the great organizing principles of geometry and mechanics—and in part, of pure mathematics—are "conventions, or *definitions in disguise*." Now, it is clear that definitions *are* "conventions": they are stipulations—or agreements, since one assumes that the stipulation will be accepted at least within a given discussion by all the discussants—concerning how a word or phrase is to be used. But this does not help us, because Adolf Grünbaum has always been quite explicit that his claims concerning conventionality are *not* about the "trivial semantic conception" of conventionality (and of course, if they were, the claims themselves would be trivial, and there would be no need for discussion).

Now, I do not know how to characterize "conventional" and its contrary in a general sense that particularizes to the one that Grünbaum has in mind in this context, in a clear way.[30] It does not follow that it is impossible to give such a characterization;

[28] Of course this slightly modified construction could have been used in Theorem 3 itself.

[29] Or: the intransigence of apparent philosophical problems.

[30] The notion of a distinction between a "merely conventional" definition and one that is not so—i.e., that *is* "conventional" *only* in the "trivial semantical sense"—seems closely related to the distinction, in traditional philosophy, between (merely) "nominal," and "real definitions"—ones that

but since I am unable to give one, I shall content myself—a little later—with suggesting what I think is a helpful explanation of what, *just in this particular context*, would count for Grünbaum (and, I suppose, for his supporters) as *not* conventional (in the non-trivial sense of "convention"). I have some hope that my explanation will be acceptable to Grünbaum, because (I say a little shamefacedly) I shall there merely copy, or paraphrase, part of what I have read in his recent paper. If I deserve any credit for this, it is only that of having at last realized that Grünbaum's claim has all along been misunderstood by those—among whom I count myself—who have objected that the Einstein–Minkowski concept of simultaneity is *not* just a "convention."

And I continue to hold this latter position; but—I now hope—in a sense that does not conflict with Grünbaum's main contention, because I am using the word "convention" with a different meaning from his.

Before I offer my new understanding of what Grünbaum has contended for all along, I shall suggest a few corrections of detail to some of his adversarial remarks, and shall try to explain the sense in which I—and, I believe, Malament and others who have disagreed with Grünbaum—have understood the issue about simultaneity.

I pointed out in [3, p. 153] that the question of "conventionality" is a different one for the procedure of Einstein in 1905 from what it is for that of Minkowski in 1907–8: Einstein was *seeking a theory* that should satisfy certain requirements—a theory that did not yet exist; whereas Minkowski was seeking the *most cogent and instructive formulation* of a theory already in existence. We, of course, are not at all in the situation of Einstein;[31] but it seems worthwhile to discuss briefly how the issue looks from the perspective of that situation—all the more, in view of the fact that Grünbaum has cited some words of Einstein in support of the conventionality thesis.

It is indeed true, as Grünbaum remarks, that Einstein [7, p. 279][32] characterizes his conception of simultaneity as the result of a *Festsetzung*, or "stipulation" by means of a definition: a definition according to which, for an observer at rest in "a coordinate system in which the Newtonian mechanical equations are valid" (*ibid.*, p. 277), the time that light takes to get from A to B is the same as the time it takes to get from B to A. This is very clearly a convention, then—but it is "clearly" so *only* in the "trivial semantical sense": it is a definition. Should one conclude that it is a convention in a nontrivial sense? It seems to me difficult to draw this conclusion simply from Einstein's own words in this passage.

Grünbaum takes Michael Friedman to task for saying that the theory that resulted from Einstein's investigation "postulates" metrical relations that include the notion of "relative simultaneity" for distant events; Grünbaum's comment [8, p. 14] is:

in some sense define the "essence" of something; it is well known that many traditional philosophers have rejected such a notion entirely: this is the "nominalist" position.

[31] Not in that of Minkowski either, since Minkowski has performed his task; nonetheless our situation more nearly resembles Minkowski's, in that we are concerned with a critical discussion of the theory.

[32] The page reference is to the *Collected Papers*.

> But, as we saw, Einstein stated emphatically that assertions of metrical simultaneity in the STR are *not* "hypotheses" which are "postulated" in Friedman's sense, ontologically on a par with, say, the postulate that light is the fastest causal chain. Why then does Friedman feel entitled to gloss over that important ontological difference by using the same term "postulate" for both?

But this ignores something else that Einstein says, *before* the passage about "stipulating" equal speeds of light in opposite directions: in the second paragraph of the introductory section of the paper, after speaking of empirical evidence that leads to the "conjecture"—*Vermutung*—that not only in mechanics but also in electrodynamics "no properties of the phenomena correspond to the concept of absolute rest, but that rather [—*vielmehr*—] for all coordinate systems for which the mechanical equations are valid the same electromagnetic and optical laws are also valid," Einstein has written (and I quote his words in German first, to make sure that no distortion is introduced by translation):

> Wir wollen diese Vermutung (deren Inhalt im folgenden "Prinzip der Relativität" genannt werden wird) zur Voraussetzung erheben und außerdem die mit ihm nur scheinbar unverträgliche Voraussetzung einführen, daß sich das Licht im leeren Raume stets mit einer bestimmten, vom Bewegungszustände des emittierenden Körpers unabhängigen Geschwindigkeit V fortpflanze.
>
> We intend to elevate this conjecture (whose content in the following will be called "Principle of Relativity") to a presupposition, and, besides, to introduce the presupposition—only in appearance incompatible with [the former one]—that light in empty space is always propagated with a determinate speed V, which is independent of the state of motion of the emitting body.

It would seem, then, that we have Einstein's authority after all for characterizing as a "postulate" (or "presupposition" or "hypothesis") the principle that the "speed of light is the same" in one direction as in the other. This of course does not decide the issue as to whether these postulates themselves should be regarded as "conventions" (as Poincaré *did* regard the axioms of geometry); it bears only on the particular appeal to Einstein's statements made by Grünbaum.

The situation, then, for Einstein's investigation was this: he *did* have reasons to want a theory that satisfies the two "presuppositions" he formulated. The urgent desirability of such a theory had been emphasized in 1900 by Poincaré [9]. When Poincaré himself *solved* this problem in 1905, he regarded the solution as a mere *tour de force* (and in subsequent writings, after the publication of his great paper [5] [see, e.g., Part 3 of *Science and Method*, which reproduces a review article of 1908], he never referred to his own work but only to the not quite satisfactory "new dynamics" of Lorentz). Whether or not he had read the 1900 report by Poincaré. Einstein was motivated by the same considerations as Poincaré; and if, both having found essentially the same theory, Einstein's *view* of it was radically different from Poincaré's, this rests to no small degree upon the fact that Einstein had subjected the concept of time to a much deeper criticism than had Poincaré, for whom the "transformed time-coordinate t'" was no more than a *mathematical trick* to make the theory work. (I believe that this is not something that Grünbaum will disagree with.)

So: the *problem* lay precisely in the "apparent contradiction" referred to by Einstein between his two "presuppositions." The *resolution* he found of this apparent contradiction consisted precisely in his realization—and it evidently cost him considerable intellectual struggle (cf. [9])—that *the discrepancy in synchronization* between the time t and the transformed time t' in the Lorentz transformation—a discrepancy that had led Lorentz to call t' the "local time"—corresponds to a *conceptual gap* concerning the concept of "simultaneity" or "synchronization": that, as Einstein says [11, p. 61], in a passage also cited by Grünbaum [8, p. 14], "There is no such thing as simultaneity of distant events."[33]—This recognition opened the way to the introduction of a suitable definition to close that conceptual gap; and it is not at all surprising, therefore, that Einstein was concerned to emphasize to his readers that *a definition was here (a) needed and (b) (therefore) legitimate*.

In the light of all this—considering the fact that it was essential for the success of Einstein's project to find a systematization of spatiotemporal relations and measures that would satisfy two requirements: (1) that for investigators who use these measures, the laws of classical mechanics and of classical electrodynamics (including optics) hold (at least to high approximation) for the results of measurement, and (2) that this should be true for a system of teams of investigators, the investigators of each single team being mutually at rest, the investigators of different teams in arbitrary states of uniform motion relative to one another; the fact that the classical laws presuppose, for any single such team of observers, a standard of synchronization; and the fact that any standard of synchronization that meets these requirements *must* agree, in application, with the criterion proposed by Einstein's *Festsetzung*—it seems somewhat misleading to call the latter a "convention" in a deeper sense than the one applicable to all matters of linguistic usage.

Just one further turn regarding this aspect of the matter—i.e., Einstein's own procedure: Einstein could perfectly well have contented himself with the *Voraussetzungen* he formulates in his introductory section, and *instead* of "defining" simultaneity, have *deduced* from these two assumptions that light that travels back and forth (or *vice versa*!) between two "inertial" observers A and B at relative rest must take equal times both ways, and therefore can serve as a signal to synchronize clocks in precisely the way the "definition" prescribes.—I do not think this alternate expository mode changes anything essential: an upholder of the view of "conventionality" could just say that the *Voraussetzungen* themselves have to be counted as "conventions," rather as Poincaré did with respect to the axioms of geometry and mechanics. And I remind my readers (and myself!) that I do not here claim to "settle" the issue of conventionality—on the contrary, I have already said that for me the very notion (in general) of what is or is not a "convention" is distressingly unclear; I should really prefer to say the sort of things I have already said about the *role played* by

[33] Grünbaum urges the fact that, according to Einstein, this is one of the "insights of definitive character that physics owes to special relativity," as showing again that Einstein is on Grünbaum's side on the question of the "conventionality" of simultaneity. But what Einstein says in the passage cited, in the very next clause, is: "thus there is no unmediated distance action in the sense of Newtonian mechanics." The point—the *"definitive insight"*—is that there is no such thing in nature as simultaneity *schlechthin* of distant events: no *absolute*, but only (at most) *relative* simultaneity. At any rate, simultaneity "relative to" either an observer or an inertial system *is not mentioned by Einstein in this passage at all!*

Einstein's concept of simultaneity, and leave the word "convention" out of the discussion entirely.

Let us then proceed to the other point of view: that of the finished theory. The theorems of the first part of this paper—together, of course, with that of Malament—are for me the main text for this point of view; but they do require some comment.

First, then, I have already said that the generalization contained in Theorem 2 does not seem to me of much philosophical interest. The reason is that the "communities" of observers who share a simultaneity equivalence-class, in the "new" case—where the classes are "Minkowski hemispheres"—are, as noted, constantly changing; it is therefore impossible to refer any sort of stable measurements to them: the *purpose* for which Einstein's systems of inertial observers were introduced has been entirely lost.

But some remarks about that purpose also need to be made. These systems of inertial observers play, for the theory, the role of a kind of Platonic myth. In actual fact there are *no* such observers: first because the real world is *not* characterized by a Minkowski space-time, and second because "even if it were," it would be very hard to see how even *one* such system of inertial observers could be created.—What on earth can the meaning of "even if it were" be here?—Well, I should say (in the spirit still of "Platonic myth-making"), we might just envisage the possibility that a theory of gravitation (such as Poincaré and Minkowski independently did attempt to formulate) could be envisaged within the Einstein–Minkowski framework. If for a moment we consider that as a possibility, the first observation to make concerning our problem is that *we* certainly are *not* (would not be) inertial observers, since we live on a body that is not in a state of inertial motion. In order to produce an inertial environment, we should have to embark upon a program of space-travel expressly designed for that purpose: that is, to devise space vehicles whose navigational systems were designed to compensate exactly for gravitational forces.—I shall not continue with this fantasy; I am not good enough at science fiction. But I hope the point will be clear—how far from "reality" these envisaged inertial observers are. They are none the less useful, however, as vivid embodiments of the relationships expressed by the "congruence transformations" of Minkowskian geometry—i.e., of the Lorentz transformations (both homogeneous and inhomogeneous—in other words, the Poincaré group). But there are two corollaries of these elementary remarks: (1) that Einstein's "definition" of simultaneity, and analogous considerations, are best thought of, not as *quasi*-"operational" definitions, but as depicting" something like "thought experiments" to make vivid the situation in this theory; (2) that it is indeed the situation "in this theory" that is concerned—*not* that "in the real world": any insight provided into "the real world" comes only through the fact that the theory can claim to give "partial," or "approximate," information—or, perhaps, information of an "infinitesimal" kind—about the real world. To put it simply: any conclusions we are inclined to draw about such things as "conventionality" or the opposite should in principle refer, in the first instance, to how things stand, conceptually, *within the theory*.

With this understood, I have one remark to make that may be surprising: it is that unlike Theorem 2, I think there *is* a point of (mild) "philosophical interest" in Theorem 3—which deals with a space-time of two dimensions, hence a space of only one dimension, which is extremely far from the "real world." This interest attaches to the easier and more familiar branch of the theorem—the case where the simultaneity equivalence-classes are straight lines. The point is that just here we find a "possible system of inertial observers" with a *deviant* concept of "relative simultaneity"; I hope it will be a little instructive to examine this case.

But just what *is* this deviant concept (or what *are these* deviant *concepts*)?—The answer is that they are very close to the ε-relations of Reichenbach, for whose viability Grünbaum has long contended; and they *coincide*—but under the drastic dimensional restriction—with a conception put forward by Allen Janis (cited by Grünbaum [7, pp. 9–10]). For if we choose a system of relative simultaneity relations in the way described in Theorem 3, the axes determined by the spatially oriented radius s (represented by a real number of absolute value less than 1, positive if "to the right," negative if "to the left"), then our "observers" are taking the ratio of "the speed of light to the right" to "the speed of light to the left" to be $(1-s)/(1+s)$; and this amounts, for a Reichenbachian observer, to choosing $\varepsilon = (1+s)/2$ for light sent, from his position, to the right, and $\varepsilon = (1-s)/2$ for light sent, from his position, to the left. (This differs from Reichenbach's principal example [12, p. 127] in that Reichenbach supposes one "central" observer who uses the same value of ε for all directions;[34] but he also gives an example [12, p. 162] in which ε depends on the direction, so our present situation does fall under the class of those he at least implicitly envisages.)

There are two reasons why this possibility *does* emerge in two dimensions but (from the point of view under discussion) does *not* in higher dimensions. One simple point that rules this out in higher dimensions is that such a choice of the relation violates the relativistic invariance principle. It would do so in two dimensions also, if (as in Malament's theorem) we required invariance under *reflections*. In higher dimensions it violates even the narrower invariance requirement, because any spatial direction can be transformed to any other by rotation (whereas when there is only one dimension of space, there is no room to turn around: we can distinguish left and right, and make the axes lean, or the speeds differ). The second reason is in a way more interesting: in higher dimensions, *there is no "intrinsic," or "objective," way to* INSTRUCT *an observer as to how to make the choice of a preferred direction.*[35]

This does not mean that such a choice could not be *made*; it only means that it could not be made *according to a "universal" rule*: a "team" of inertial (imaginary!—science-fictional!) observers, at rest with respect to one another, who wished to carry out systematic measurements and to determine (for instance) velocities and accelerations using a "Janis-simultaneity relation" would have to come to a special agreement with one another as to how to determine the direction of the

[34] This is not really clear from Reichenbach's text at this point; at least, it has not been clear to *me*: I had until recently always supposed that Reichenbach wanted the speed of light to be constant in a given direction—and this would obviously necessitate, for $\varepsilon \neq 1/2$, that ε be different in different directions.

[35] It might be argued that this "more interesting" point is really the same one as the first point; this bears on the question of the connection of "invariance with respect to" and "definable *from*"—on which there will be a little more below.

world-line that is to be their axis of simultaneity; and this agreement would have to use some special features of the "geography"—or, rather, the "cosmography" of their universe. This would in general not be an easy thing to do; its possibility would depend on the existence of recognizable, and *stable*, features of their cosmos to serve as (the analogue of) landmarks for determining and *redetermining* directions that "are the same" for all the observers and "remain the same" over time. In the alternative choice of a simultaneity relation described by Janis [8], the person who chooses this alternative notion does so "by specifying a set of three parameters," and thereby singling out a time-like direction that is inclined to the investigator's own. The three parameters required are the coordinates—in the projective space associated with space-time—of a time-like direction: three are needed because the projective space of the directions in a two-dimensional affine space is three-dimensional. *But* one can single out a direction in that way only if *coordinates have been laid down* for that space of directions. Since this is itself a task at least as complicated as that of determining a relation of simultaneity, to speak so cursorily of "specifying three parameters" partially masks the problem.—Note that the problem lies, not in the need to specify *the values of the parameters*, which, as real numbers, are available as "individual concepts" belonging to the logico-mathematical apparatus, but in the need to choose a *way of relating parameters to directions* (i.e., a "coordinatization" of the projective space). That is why the problem does not arise for a two-dimensional Minkowski space-time: the "space of directions" of such a space-time is one-dimensional, which means that there is an "intrinsic" association of directions with real numbers.

An analogy may help to make the main point clear. When *temperature* was first introduced into physics, and first *measured*, the quantity so named was not a single one at all—there were as many such "quantities" as there were types of thermometer, and the choice among them was *nothing but* "conventional." Indeed, for various investigators—or the same investigator in various experiments—who were concerned with temperature, whether the quantities they referred had simple relations to one another depended entirely upon stability in this respect; the quantities could "in principle" vary with the thermometric material, with the material in which (if the thermometric material was a gas or liquid) that material was contained, in the proportions of the containing vessel and of its cavity, etc. The situation became radically different after the development of the second law of thermodynamics: now a *theoretical definition* of temperature was available, that determined the quantity "temperature" precisely, leaving open only the choice of a unit for the temperature scale: the notion of *ratios of temperatures*—which, initially, would have seemed *least* likely to have any significance *at all* (since the zero-point of a temperature scale was at first entirely arbitrary)—had now received an "absolute" theoretical meaning.

It was perhaps—it *is* perhaps—still open to a disputant to argue that the so-called "absolute temperature" itself remains a matter of convention, chosen in the interest of "merely descriptive simplicity." I should not care to debate that point. But I *do* maintain, and think it important to recognize, that the difference, within the special theory of relativity, between the "simultaneity relative to a state of inertial motion" of Einstein and Minkowski, and the simultaneity relations described by Janis or by Reichenbach, resembles in an important way the difference between the "material" conceptions of temperature as a quantity, and the "absolute" conception offered by developed thermodynamics.

Before coming specifically to Grünbaum's claims and the misunderstandings—on both sides—that I now believe to have muddied the issue, I want to mention one interesting point raised by Sarkar and Stachel. They speak of the possibility of "formulat[ing] the basic structure of the special theory of relativity without the use of any simultaneity convention" [4, p. 219]. This *is* certainly possible—there is more than one way to interpret their words; but in the strongest way of all, namely taking them to mean "without the use of any conception whatever of "distant simultaneity," it is surely possible. Indeed, the *general* theory of relativity, in its "most general case," altogether lacks any such notion as distant simultaneity; and this does not prevent the theory from being formulated. But: (1) A formulation that dispenses with any use notion of relative simultaneity must also dispense with any notion of relative velocity, and with the notion of acceleration in its usual form. Therefore, (2) such a formulation is not adapted to the comparison of the theory with Newtonian mechanics. On the other hand, (3) such a comparison is instructive; and since, as we know, it can be made, it must follow that (4) the formulation of special relativity that does not *use* a notion of relative simultaneity must nonetheless include the means of formulating such a notion whenever it is *desired* to do so. The reason a notion of distant simultaneity is not needed to formulate physical laws is that physical interactions, in this theory, are "infinitesimally near-by" interactions, governed entirely by partial differential equations. There *is* an "infinitesimal counterpart" of "simultaneity equivalence-class," relative, *not* to an "inertial observer," but to any state of (smooth) motion, inertial or not: it is the space-like hyperplane, in the tangent space to space-time, that is orthogonal to the tangent-line of the world-line of that motion. This infinitesimal (or "differential") notion is quite indispensable, both in the special and in the general theory. In the special theory, for the special case of "inertial motions that constitute, together, a state of relative rest," this "infinitesimal" notion is (uniquely) *integrable*; and *that* is the description of the Einstein–Minkowski relative simultaneity concept that, in my own view, presents the best case for its "true standing"[36] in the theory.

Turning, then (at last!) to Grünbaum's views as I now understand them, I have first to complement my earlier discussion of the notion "definable from" or "definable in terms of"; for this is one of the phrases in which it seems clear that people on each side of the debate have misunderstood their opponents' statements. For Grünbaum's part, he has certainly used that phrase in a sense very different from that of its customary use in logic and mathematics. This is not an intellectual crime—but it *is* a misfortune for all parties, when one of them uses a term in an unusual way without taking pains to explain this fact (of course, such an occurrence is not deliberate: it results from the fact that the party in question does not realize what the customary usage is). I was mystified to read that Grünbaum [7, p. 3] *rejected*

[36] Do I mean its "non-conventional" standing?—I have said that I should really prefer to express my opinions without using the word "conventional" at all.

Malament's condition, on a relation "definable in terms of the relation of causal connectibility," that it be invariant under all "causal automorphisms"—that is, one-to-one mappings of space-time to itself that *preserve* the relation κ of causal connectibility. I have explained above, in discussing the paper of Sarkar and Stachel, the grounds for this condition (although not long ago I should have thought this something too well known to require "explanation"). But *part* of the clarification of Grünbaum's use of the term appears when he attributes to Bas van Fraassen the remark that Malament defines the notion he is defending "in terms of κ *alone*" [8, p. 11, emphasis in the original]. I call this "part of" the clarification, because it shows that Grünbaum does not mean the same thing by "definable from…" as by "definable from… alone"; this still leaves us with the question, which for some time seemed to me hard to answer, what he *does* mean by the former expression. If, for example, one asks a geometer whether the concept of "the vertical direction" can be defined "from" the basic concepts of Euclidean geometry, the geometer would surely say no: "vertical" is, first of all, a concept of physics, not of geometry; and second, it is only defined for points near the earth's surface—or, in a more sophisticated view, for points at which there is a non-vanishing gravitational field. But it seems that in Grünbaum's usage the answer to that question must be yes: "vertical" means "in the direction of the gravitational field," and the notion of "direction" used here can be—ordinarily *is*—that based upon Euclidean geometry.—I may be wrong here in my interpretation of Grünbaum; if so, I am willing to be corrected. This of course still does not answer the question that I have said remains open; I shall try—again, under correction—to give at least a partial answer presently.

At any rate, Grünbaum has surely misread Malament when he writes of the latter as follows: "But before giving [[the]] proof, [[Malament]] declared: 'To be sure, there are other two-place relations [of relative simultaneity] which are definable from κ and O [i.e., relative simultaneity relations corresponding to non-standard synchrony, for example, some fixed $\varepsilon \neq 1/2$]. But all these are ruled out if minimal seeming innocuous conditions are imposed.'" The words in double brackets are substitutes by me for Grünbaum's words—substitutions made only to adapt the passage from its context in Grünbaum to the context here; the passages in single brackets are bracketed in Grünbaum's own text, and are interpretations offered by him of Malament's words. They are serious misinterpretations: (1) The other two-place relations Malament means are ones that are quite unsuited to serve as relative simultaneity relations—they are the relations that are ruled out by his conditions that a simultaneity relation S relative to an observer O be an equivalence relation; that it hold between some point on the world-line of O and some point not on that line; and that it not be the universal relation. (2) In particular, not only (as stated in (1)) are the excluded relations not relative simultaneity relations at all, but the Reichenbachian $\varepsilon \neq 1/2$ relations are *not* examples of relations "which are definable from κ and O." This strange misconception must have arisen from the fact that *after* discussing the matter of "definability from κ and O," Malament lists invariance as the first of his conditions on S: Grünbaum has failed to realize that Malament has done so for the sake of the exact mathematical formulation of his result, *not* because invariance is a special condition *added* to "definability from κ and O"; on the contrary, it is relations that satisfy this invariance condition, and *only* these, that Malament has called "definable from κ and O."

All this is pedantry; necessary, I think, but not edifying. Let us now consider what, in my present opinion, is the sound core of Grünbaum's view. In discussing this, I am going to have to dissent (still) from some of Grünbaum's particular *expressions* of that view; and to begin with, from this one, which initially puzzled me as much or more as did the one concerning definability: Grünbaum asks whether—and clearly means to deny that—"the facts of causal connectibility and non-connectibility *mandate* (dictate)" the standard (Einstein–Minkowski) relation of simultaneity relative to an inertial frame [8, p. 2; cf. pp. 3 and 5 for the explicit denial that this relation is "mandated" by those facts]. I was, and remain, still *more* puzzled by his claim that the standard relation of relative simultaneity, unlike the relation of causal connectibility, lacks a *fundamentum in re*: a "foundation in the thing," or "in nature" (pp. 12–13); and that "there is no fact to the matter" in ascriptions of this relation (pp. 1, 9, 12, 13)—that they lack "facticity"[37] (pp. 2, 3, 12). What is very strange here, in point of the "foundation in nature" especially, is this: we *know* that the whole metric structure of Minkowski space-time (without a distinguished spatio-temporal unit) is definable from the one basic relation of causal connectibility (the symmetric one if a time-orientation is not presupposed, the asymmetric one if such an orientation is presupposed). Therefore this structure—and in particular, the relation of orthogonality, which gives the Einstein–Minkowski relative-simultaneity relation—has a "foundation" in the relation κ of causal connectibility (again: symmetric or asymmetric). So if this relation has a "foundation in the thing," the standard relation of simultaneity relative to an inertial state has one also—assuming that the relation "*A* has a foundation in *B*" is transitive; which seems hard to deny. By the same token, it would seem that the ascription of the relation of relative simultaneity has "factual content." And as to being "mandated": a relation *"founded on"* κ would seem to derive whatever is meant by a "mandate" from that fact itself, as long as κ is regarded as "founded in things."

Of these puzzles, the one about having a foundation seems to me irresoluble: I may be wrong, but I think that Grünbaum has simply overlooked what I just referred to as the transitivity of "foundedness" (probably because he has not quite seen the importance, and the *strength*, of "definable from" in what I have called the usual sense). But the other two puzzles seem to me to have a solution. It is easiest to see with the question of factual content. Suppose I say that the space-time vector pq is orthogonal to the time-like straight world-line O. I maintain—since I have argued that the relation of orthogonality (and also the properties of linearity and time-like-ness, since they belong to the geometry that is derivable from the relation κ) is (are) "founded in things"—that that statement conveys factual content (always, of coursed, assuming, contrary to fact, that the world *is* Einstein–Minkowskian). I hope that I have said enough to persuade Grünbaum that this is so. What I think he will *deny* is that this admitted factual content is *about simultaneity*; so if I say that p and q are simultaneous relative to O, Grünbaum will deny that *this* statement

[37] Linguistic point: this word seems like the nominalization of the adjective "factitious," whose meaning is opposite to the one desired; I should suggest "factuality."

has factual content—even though I myself have *defined* "simultaneity relative to O" to be just the relation expressed by the former statement. In other words, what—according to Grünbaum—what the theory, or the facts of causal connectedness, does not "mandate" is *how the word simultaneity is to be used*.

Now, if this is all there is to it, it looks as if Grünbaum is after all defending nothing more than the "trivial semantic conventionality" of the use of the words "simultaneous" and "simultaneity." But I think that is not quite all there is to it. There is a certain traditional baggage carried by the word "simultaneous"; and I think Grünbaum is *rightly* maintaining that that baggage has no place in the special theory of relativity. But this statement is crudely metaphorical; is it possible to say clearly what this "baggage" is?—Probably not; but I think it was a mistake on the part of Wittgenstein when he produced his celebrated aphorism that whatever can be said at all can be said clearly: my own motto is that whatever one thinks is worth saying, one should try to say as clearly as one can. An *exact* statement *is* possible, and I shall make it; it is the one about which I expressed "some hope" that Grünbaum will accept it (as far as it goes). This exact statement, however, will not express what the "baggage" is, but only clarify something about the source of the latter. I shall then try to indicate—but only by indirection and example—something about what baggage the relativistic notion of simultaneity *does not* carry.

The exact statement is based upon Grünbaum's own very clear depiction of the state of affairs in pre-relativistic theory and in the special theory of relativity—that is, in the Newtonian world and that of Einstein [7] and Minkowski. I paraphrase what he has said thus: In each of these theories of physics, there is an "objective" division of the world, viewed from any point p, into the class of points that are *past* for p, those that are *future* for p, and those that are neither.[38] An admissible time-function[39] is a function τ on the entire space-time of the theory such that for any pair of points p, q, if p is in the past of q then $\tau(p) < \tau(q)$. If τ is a time-function, then the relation between p and q expressed by $\tau(p) = \tau(q)$ is an admissible simultaneity-relation. In the Newtonian theory there is a great abundance of admissible time-functions, but there is a *unique* admissible simultaneity-relation; in the Einstein–Minkowski theory, this is very far from true. In so far as any admissible time-function, and correspondingly any admissible simultaneity-relation, is compatible with the structure of the Einstein–Minkowski theory, the choice among them is a matter of "convention" (or, for that matter, "convenience"); in Grünbaum's terminology, all these functions, and all these relations, are "definable in terms of" the structure of Einstein–Minkowski space-time. An example of a space-time structure

[38] It is crucial, in Grünbaum's opinion, that this classification is grounded in facts about "causal connection." I am a skeptic in this matter (not a disbeliever, an agnostic). I agree that the notions of "causal past" and "causal future" are deeply important in the theories—perhaps especially so in the theory of relativity; but, on the other hand, (1) I think the notion of "cause" itself is in some degree problematic, so that what [if anything!] we "know" about this notion is derived from the knowledge we have gained in physics, and is not the "foundation" of the latter; (2) in assessing the knowledge we have from physics we cannot ignore quantum physics; and (3) how quantum physics ultimately affects the framework notions of relativity theory seems to remain a problem.

[39] My own term—this is intended as my own *paraphrase*, or formulation of what I believe *follows* from what Grünbaum has said.

"in terms of which" *there is no "definable" simultaneity-relation* is that of Gödel: in the original Gödel "rotating universe," there is in fact no admissible time-function whatever.[40]

But why should we accept this very liberal notion of what is "admissible," not alongside of, but *instead of*, the narrower—stronger!—notion of what is (in the usual sense) "definable from" the structure of Newtonian space-time on the one hand, Einstein–Minkowski space-time on the other?—Indeed, I do not think we *should* "accept [the former]… instead of [the latter]"; that is what most of this paper has been about. I think we should acknowledge, side by side, the points made by Grünbaum and the points made by Malament et al.

As to the matter of "baggage": I have already stated my objection to Grünbaum's claim that the Einstein–Minkowski notion lacks a *fundamentum in re*; but the baggage I have referred to *is* "metaphysical baggage." I suspect—I confess that I *hope*—that at least part of what Grünbaum means in rejecting *any* "metaphysical foundation" for relative simultaneity is what I myself meant, long ago, when I wrote the following, in criticizing metaphysical arguments of C. W. Rietdijk and of Hilary Putnam:

> [W]hat Einstein's arguments showed was that *a certain procedure of measurement singles out a time axis and gives numerical time differences dependent upon that distinguished axis*; not that an observer's state of motion imposes upon him a special view of the world's structure. This illegitimate metaphysical interpretation of the time-coordinate appears perhaps most plainly in Rietdijk's phrase describing *C* and *A*, when at rest with respect to one another, as "experiencing the same 'present"'; there is of course no such "experience": the fact that there is no experience of the presentness of remote events was one of Einstein's basic starting points [13, p. 16, n. 15].

The "baggage," then, can be said to be the carrying around of a special relation of simultaneity, as it were "in one's head." I believe that A. N. Whitehead thought something like this, when he contrasted, among "actual entities," the relation of "causal efficacy" and that of "presentational immediacy": in the latter, the mode of *perceptual space*, what we perceive is the entire present simultaneity slice "relative to us," *as if* it were characterized by the perceptual qualities that we experience. Perhaps I am wrong about Whitehead; at any rate, it is an impossible conception. If there were no other trouble with it, what are we to say about an observer who is *not* in a state of uniform motion? For such an observer, it is entirely possible—indeed, it is certain!—that "his or her" simultaneity-slice at one moment will contain "events" that are in the *future* of "his or her" simultaneity slice at a *later* moment; a perfect muddle! It is, then, certainly *not* the case that the special theory of relativity "mandates" that a sentient being carry such a relation around through that being's career. A paradigm of what I think of as the *poignancy* of the "old" notion of simultaneity that is quite lost to the new one is the old sentimental song-line, "I wonder who's kissing her now!"—In fact, for events that are within normal human spatio-temporal range of one another, the special (or the general) theory of relativity

[40] This, then, clarifies, at least to some extent, what Grünbaum's notion of "definable in terms of" can *exclude*.

provides a perfectly intelligible notion of "now" to *carry* that kind of poignancy; and it is *not* the *geometrical* notion of the "instantaneous now" relative to a state of inertial motion.[41]

The moral that I draw, then, is that although Grünbaum is (in my opinion) wrong to believe that, in so far as the causal theory of time has real—or factual—or fundamental content, the Einstein–Minkowski notion of relative simultaneity does *not* have such content, he is right to deny that this notion has content entirely comparable to that of the old Newtonian relation of absolute simultaneity. I do not believe that Malament, for one, would differ with Grünbaum on this point any more than I do.

3 Supplementary Notes

1. Hogarth's proof is far more complicated than the theorem requires; here is a simpler one: What is to be proved is:

   If to every "inertial observer world-line" (more briefly: "observer-line") O there is assigned an equivalence relation between space-time points, "p and q are simultaneous for O," (a) invariant under all maps of Minkowski space onto itself that preserve the Minkowski quadratic form, and such that (b) for every point p there is a unique point q on O that is simultaneous with p for O, then points p, q, are simultaneous for O if and only if they lie in a hyperplane orthogonal to O.

Proof. First, let p and q be simultaneous for O. Let p_0 be the point on O that is simultaneous with p—and so also with q—for O, and let h be the hyperplane through p orthogonal to O. Reflection of space-time in the (space-like) hyperplane h is a map satisfying the conditions laid down; under it, p is fixed and O is mapped to itself; so, by the invariance condition (a), the image p_0' of p_0 is simultaneous with p for O. Unless p_0' and p_0 coincide—i.e., unless p_0 is in the hyperplane h—this implies that two distinct points, p_0 and p_0', both on O, are simultaneous with p for O. Since this violates condition (b), p_0 must lie in h. But q satisfies the same conditions as p, *vis-à-vis* O and p_0; p_0 therefore lies also in the hyperplane through q orthogonal to O. Since p_0 lies in only only one hyperplane orthogonal to O, and this is h, q too must lie in h. This establishes the "only if" clause of the theorem. Second, if h is a hyperplane orthogonal to O, and if p and q lie in h, let p_0 be the point of O simultaneous, for O, with p; then by what we have already seen, p and p_0 lie in a hyperplane orthogonal to O—and this can only be h. By the same token, q must be simultaneous, for O, with a point of O that lies in h—and this can only be p_0. It follows that p and q, since each is simultaneous for O with p_0, are themselves simultaneous for O; and this completes the proof.

2. As stated above (end of n. 6), the formulation of the theorem on p. 149 of [3] not only fails to state accurately what the argument preceding it has established, but is simply false. A correct statement is:

[41] What it *is*, is discussed, implicitly, in Stein [3, p. 159].

If R is a reflexive, transitive relation on a Minkowski space (of any number of dimensions—of course at least two), invariant under automorphisms that preserve the time-orientation, and if Rxy does not hold for every pair of points (x, y) of the space, but does hold whenever xy is a past-pointing (time-like or null) nonzero vector, then Rxy holds if and only if xy is a past-pointing vector.

If the dimension is greater than two, the automorphisms considered may be restricted further to such as preserve the *spatial* orientation as well the time-orientation [equivalently: that preserve the orientation of the whole manifold as well as the time-orientation], and also preserve the scale.

3. In the proof—or proof sketch—they give for their Theorem 1 (which is essentially the same as case (b) of Theorem 1 above), the exposition of Sarkar and Stachel is not at all points quite clear: for instance, they refer, near the beginning of part (ii) of their argument [4, p. 217], to "the family of hypersurfaces of simultaneity," although they have not given any reason to suppose that the equivalence-classes of the simultaneity relation *are* hypersurfaces (the counterexamples given above show that in the absence of a requirement of scale-invariance this need not be the case), or even that the equivalence-classes *contain* hypersurfaces; so one cannot be entirely sure exactly what they may be assuming tacitly. Nevertheless, there *is* in their proof one passage containing a clearly identifiable and crucial paralogism. They have (almost) correctly remarked, at the beginning of (ii), that "[a]ccording to our definition, any simultaneity relation causally' definable from κ and O must be invariant under any transformation belong to the group of O causal' automorphisms. This implies that it must take the family of hypersurfaces of simultaneity onto itself under any such automorphism" (*sic*; but read, of course, "that any such automorphism must take [etc.]"). Some lines below this, however, they say of translations orthogonal to the world-lines of the inertial system, "If they are not to affect the simultaneity relation (which amounts to our assumption that the simultaneity relation is independent of the initially-chosen world line O), these translations must take each simultaneity hypersurface onto itself." This simply does *not* follow: what does, is just that each translation must take each equivalence-class to *some* equivalence-class, not necessarily to *itself*. The point is crucial, because it is only from the premise that each equivalence-class (a) is a hypersurface, and (b) is mapped *to itself* by translations orthogonal to the inertial system, that they conclude that the classes are hyperplanes orthogonal to the inertial system. Indeed, the mere assumption of *translation*-invariance, if one did not also postulate invariance under *rotations* that take the inertial system to itself, would allow the possibility of simultaneity hyperplanes "*inclined at a fixed angle*" to the world-lines of the inertial system; the system of such hyperplanes would, then, be invariant under translations orthogonal to the inertial-system's world-lines; but the *individual* hyperplanes would *not* be invariant under these translations. And Sarkar and Stachel—as they seem not to have noticed!—not only make no appeal to scale-change-invariance, they likewise make no appeal to rotation-invariance in their proof-sketch.

I have not discussed Theorem 2 of Sarkar and Stachel. It is fairly clear what this theorem is intended to say, and that what it is intended to say is true. But its formulation [4, p. 218] is, when one looks at it closely, very obscure; and the proof given for it there is garbled. As to the obscurity: in this theorem, the authors impose the condition—condition (ii)—that "no event is simultaneous with one in its causal future (past)"; the parenthesis is intended to imply an *alternative*: one of the two simultaneity relations they arrive at has as its equivalence-classes the "backwards" mantle of a null-cone, the other the "forwards" mantle; the former contains "no event in the causal future" of the vertex of the cone, the latter "no event in its causal past." But the condition as formulated certainly does not do what it is meant to: simultaneity is—not just usually, but explicitly for Sarkar and Stachel—an equivalence relation. If event e is simultaneous with e', and if e' is on the backwards mantle of the cone of e and thus "not in the causal future of e," then *ipso facto* e' is simultaneous with e; but e is in the causal future of e'; so the condition as the authors have stated it *rules out* the mantles of the null-cone (forwards *or* backwards) as simultaneity-classes. As to the proof: The theorem requires that simultaneity be relativized to an inertial observer-line O, and that it be invariant, for every point e of O, under "boosts" at O; this is condition (i) of the theorem. The proof sketch begins: "Let p be any event not on O that is simultaneous with e. Consider the vector ep. By condition (i) of the theorem, under boosts at e, the length of this vector must remain invariant."—This, I say, is garbled: the conclusion has nothing to do with condition (i) of the theorem; "boosts," which are among the transformations in the Poincaré group, *ipso facto* preserve the lengths of vectors.—The proof continues: "Thus, the locus of p under all such boosts is either the forward or backward null cone or a time-like hyperboloid within the null cone."[42]—Again, this is just a fact about the geometry of boosts; so far, none of the conditions of the theorem has been actually used. The remainder of the proof is: "Now, e does not belong to any such hyperboloid. Therefore, if such a hyperboloid were used to define the simultaneity relation, e would not be simultaneous with itself violating the reflexivity condition of an equivalence relation. Thus, only the two half null cones remain as potential hypersurfaces of simultaneity. Condition (ii) restricts us to one of the two."

Well, we have already seen that condition (ii) cannot be helpful as it stands. But as to the connection with condition (i): the conclusion that the locus of the point p under boosts is either a half-cone or a lobe of a "hyperboloid," as already remarked, is independent of condition (i); what that condition does now imply is that this locus consists entirely of points simultaneous with e. However, what we need for the conclusion drawn by Sarkar and Stachel is that these are the *only* points simultaneous with e; and condition (i) does *not* imply this, without *some* further stipulation.

Perhaps it was overstating the matter to say that it is "fairly clear" what the theorem is intended to say. Here is an attempt at it: If we require of a simultaneity relation relative to an observer-line O (a) that for every point e on O, simultaneity is invariant under boosts at O and (b) that no observer-line meet any of the equivalence-classes of this relation in more than one point, then the only possibilities are the two Sarkar

[42] "Hyperboloid," of course, in the usual Euclidean model of a Minkowskian space-time. "Time-like" is perhaps misleading: this is a hyperboloid of two branches; vectors from one to another point of one branch are space-like; it is the separation between the two branches that is time-like.

and Stachel describe.—*This* is true; and it follows by a straightened-out redaction of their argument (as shown just above, one concludes that the class of events simultaneous with a given e on O is either one mantle of the null-cone at O, or a "branch of a hyperboloid" [the full cone or hyperboloid is ruled out by the fact that there are observer-lines that meet them in two points]; but this class must contain e—by reflexivity—whereas O meets the hyperboloid-branches in points other than e, and it must also contain e; but meeting an equivalence-class in more than one point has been excluded).

Second thoughts (or afterthought)—an alternative reconstruction of the intent of Theorem 2: it may be that condition (ii) was intended to mean that no event *on O* is simultaneous with an event in its causal future (alternatively: its causal past). This would do the trick (although the part of the proof invoking this condition would need a little rewriting).

4. In the two-dimensional case, if the equivalence-classes are "Minkowski semicircles" and if the axes are not normal to these curves, the configuration of a given equivalence-class Σ and its axes may be described as follows: The equivalence-class, as we know, has a given temporally oriented radius r. Let us represent this simply by a real number (positive or negative, denoting "future-pointing" or "past-pointing"—zero is not a possibility). In the usual Euclidean model of a Minkowski plane, Σ is a connected branch of a hyperbola whose principal semi-axis is r—understanding the sign of r to mean (taking the time-axis to be vertical) the "upper branch" if r is positive, the "lower branch" if r is negative. We may, as usual, take the center of the hyperbola—which represents the center of the "Minkowski semicircle"—to be at the origin of the system of coordinates. Now let there be given also a spatially oriented non-zero radius d (*also* representable as a real number—with an analogous convention about the sign: e.g., positive "to the right," negative "to the left". Consider a second "Minkowski semicircle," or hyperbolic branch, Ω, having the same center as Σ, but with the space-like oriented radius d; and take the axis of the equivalence-class Σ, at any given point p, to be *the line through p that is tangent to* Ω; so the family of all axes of Σ is just the family of all lines tangent to Ω: it is this that takes the place of the family of all lines through the center (which may indeed be considered as the limiting—degenerate—case of our "hyperbolic" construction when the spatial radius d goes to zero).—The radius d of the auxiliary hyperbola Ω is *not* the same as the spatially oriented radius s used in the construction described in Theorem 3 of the text above; r being given, the connection between the s and d (this is the one point that does necessitate a little calculation to determine), if we represent s, d, and r by real numbers, is given by the pair of equations, inverse to one another: $d = -sr/\sqrt{(1-s^2)}$, $s = -d/\sqrt{(d^2+r^2)}$. (Note that s must, as previously specified, be chosen with absolute value less than 1; this is guaranteed by the second equation. On the other hand, d is entirely arbitrary.—Note too that these equations also hold in the "degenerate" case (or perhaps, rather, the *normal* case!) $s = d = 0$.

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