ON THE HYPOTHESES, WHICH LIE AT THE BASIS OF GEOMETRY

Bernhard Riemann

PLAN OF THE INVESTIGATION.

As is well known, geometry presupposes both the concept [Begriff] of space, and the first principles [Grundbegriffe] for the constructions in space, as something given. It provides only nominal definitions of these, whereas the essential specifications appear in the form of axioms. The relationship of these presuppositions is left thereby in the dark: one neither sees clearly whether and to what extent their combination is necessary, nor a priori whether it is possible.

From Euclid up to Legendre (to name the most celebrated modern writer on geometry), this darkness has been lifted neither by the mathematicians nor by the philosophers who have concerned themselves with these matters. The reason for that failure surely consists in this, that the general concept of multiply extended magnitudes—under which concept the spatial magnitudes are comprised—has not been treated at all. I have therefore set myself first the task, to construct the concept of a multiply extended magnitude out of general magnitude-concepts. It will turn out that a multiply extended magnitude is susceptible of various measure-relations—so that space is but a particular instance of a triply extended magnitude. But it then follows necessarily that the propositions of geometry cannot be derived from general magnitude-concepts; that, rather, those properties by which space is distinguished from other conceivable triply extended magnitudes can only be taken from experience. Hence arises the task, to seek for the simplest facts by which the measure-relations of space can be specified—a task that from the nature of the case is not fully determinate: for several systems of simple facts can be cited, which suffice for the specification of the measure-relations of space: most important for the present purpose is that laid down by Euclid. These facts are, like all facts, not necessary, but only of empirical certainty—they are hypotheses; one can thus investigate their probability—which, of course, within the limits of observation, is very great—and subsequently judge the reliability of their extension beyond the limits of observation, both on the side of the immeasurably great and on the side of the immeasurably small.

'experiential' space? ' euclidean' space?

lovely

I. CONCEPT OF AN n-TUPLY EXTENDED MAGNITUDE.

In now attempting first to solve the first of these problems, the development of the concept of multiply extended magnitudes, I believe that I may so much the more claim an indulgent judgment as I am little practiced in such works of philosophical nature, where the difficulties lie more in the concepts than in the construction; and, with the exception of a few quite brief indications which Herr Geheimer Hofrat Gauss has given in the second memoir on biquadratic residues in the Göttingen Scholarly Notices and in his jubilee paper, and a few philosophical investigations of Herbart, I could make use of no previous work at all.

1.

Magnitude-concepts are possible only where there occurs a general concept allowing various modes of specification. According as there does or does not exist a continuous transition from one to another of these modes of specification, they constitute a continuous or a discrete manifold; the individual modes of specification are called in the first case points, in the second elements of this manifold. Concepts whose modes of specification constitute a discrete manifold are so frequent, that for arbitrarily given things there can always be found —at least in the more cultivated languages—a concept under which they are comprised (and the mathematicians have therefore been able to proceed without scruple, in the theory of discrete magnitudes, from the postulate that given things are to be considered as homogeneous); in contrast, the occasions to form concepts whose modes of specification constitute a continuous manifold are so rare in ordinary life that the places of sense-objects, and the colors, are perhaps the only simple concepts whose modes of specification constitute a multiply extended manifold. More frequent occasion for the creation and elaboration of such concepts is found only in the higher mathematics.

role of experience here?

Definite parts of a manifold, distinguished by a characteristic or a boundary, are called quanta. Their comparison in respect of quantity is effected for discrete magnitudes by counting, for continuous ones by measurement. Measuring consists in a superposition of the magnitudes to be compared; it therefore requires a means of transporting the one magnitude as a measuring-rod for the other. Failing this, one can compare two magnitudes only when the one is a part of the other—and even then can only decide the more-or-less, not the how-much. The investigations that can be instituted concerning them in this case form a general part of the theory of magnitudes, independent of measure-determinations, in which the magnitudes are regarded, not as existing independently of position and not as expressible in terms of a unit, but [just] as domains in a manifold. Such investigations have become a requisite for several parts of mathematics, especially for the treatment of multiply-valued analytic functions; and their lack is surely a principal cause why the famous theorem of Abel and the results of Lagrange, Pfaff, Jacobi have remained so long unfruitful for the general theory of differential equations. For the present purpose it suffices to bring forward from this general part of the theory of extended magnitudes—in which nothing more is presupposed than what is already contained in the concept of such magnitudes—two points: the first concerns the generation of the concept of a multiply extended magnitude; the second concerns the reduction of the specifications of place in a given manifold to specifications of quantity, and will make clear the essential criterion of an *n*-fold extension.

2.

If, for a concept whose modes of specification constitute a continuous manifold, one proceeds in a definite way from one mode of specification to another, the traversed modes of specification constitute a simply extended manifold—the essential criterion of which is that in it a continuous passage from a point is possible only in two directions, forwards or backwards. If, now, one thinks of this manifold as passing over into another entirely different one, and once again in a definite way (i.e., so that each point in one goes over to a definite point of the other), then all the modes of specification so obtained constitute a doubly extended manifold. In similar fashion one obtains a triply extended manifold if one represents to oneself a doubly extended one as passing over in a definite way into an entirely different one; and it is easy to see how this construction can be continued. If, instead of regarding the concept as subject to specification, one thinks of its object as variable, then this construction can be described as the synthesis of a variability of *n* + 1 dimensions out of a variability of *n* dimensions and a variability of one dimension.

ssuming
mply
onnected
manifolds

3.

I shall now show how, conversely, one can decompose a variability whose domain is given, into a variability of one dimension and a variability of fewer dimensions. To this end, think of a variable piece of a manifold of one dimension —reckoned from a fixed origin-point, so that its values are mutually comparable —which has for each point of the given manifold a definite value, varying continuously with that point; or in other words, assume within the given manifold a continuous function of place, and indeed a function that is not constant along a part of this manifold. Every system of points on which the function has a constant value then forms a continuous manifold of fewer dimensions than the given one. These manifolds pass continuously into one another as the function varies; it can be assumed, therefore, that one of them gives rise to all the others—and this, speaking in general, can occur in such a way that each point goes over to a definite point of the other; the exceptional cases, whose investigation is important, can be left aside here. By this means the specification of place in the given manifold is reduced to a magnitude-specification and a specification of place in a manifold whose extendedness has lower multiplicity. Now it is easy to show that the latter ~~this~~ manifold has $n - 1$ dimensions, if the given manifold is an n-tuply extended one. By n-fold iteration of this process the specification of place in an n-tuply extended manifold is therefore reduced to n magnitude-specifications; and thus the specification of place in a given manifold is reduced—if such reduction is possible—to a finite number of specifications of quantity. Nevertheless, there are also manifolds in which the specification of place requires not a finite number, but either an infinite series or a continuous manifold of magnitude-specifications. Such manifolds are constituted, e.g., by the possible specifications of a function on a given domain, the possible shapes of a spatial figure, etc.

i.e., assuming regular scalar f^ns

R does this forestall Cantor, or is it just a misstatement?

↓

or does he mean that one cannot ~~pick out~~ 'identify', i.e. a scalar f^n on a manifold w/o 'specifying' its values at all (possibly uncountably) places?

II. MEASURE-RELATIONS OF WHICH A MANIFOLD OF n DIMENSIONS IS SUSCEPTIBLE, ON THE PRESUPPOSITION THAT ITS LINES—INDEPENDENTLY OF POSITION—POSSESS A LENGTH, SO THAT EACH LINE IS MEASURABLE BY EACH.

Now that the concept of an n-tuply extended manifold has been constructed, and its essential criterion has been found to be the reducibility of the specification of place to n magnitude-specifications, there follows as the second of the tasks proposed above an investigation of the measure-relations of which such a manifold is susceptible, and of the conditions that suffice for the specification of these measure-relations. These measure-relations admit of investigation only in abstract magnitude-concepts, and of systematic representation only through formulas; yet under certain presuppositions one can resolve them into relations that are individually susceptible of a geometric representation, and thereby it becomes possible to express the results of the calculation geometrically. To win to solid ground, therefore, an abstract investigation in formulas is indeed unavoidable, but the results of the investigation will admit of a presentation in geometric garb. The foundations in both respects are contained in the celebrated memoir of Herr Geheimer Hofrat Gauss on curved surfaces.

1.

Measure-determinations require that magnitudes be independent of place, and this can occur in more than one way; the first assumption that suggests itself, and the one I intend to pursue here, is that the lengths of lines are independent of position, hence that each line is measurable by each. If the specification of place is reduced to magnitude-specifications, and thus the position of a point in the given n-tuply extended manifold is expressed by n variable magnitudes x_1, x_2, x_3 and so forth to x_n, then the specification of a line will come to this, that the magnitudes x are given as functions of one variable. The problem is then to establish a mathematical expression for the lengths of lines—to which end the magnitudes x must be considered as expressible in units. I shall treat this problem only under certain restrictions, and in the first place restrict myself to lines in which the ratios among the magnitudes dx—the respective changes of the magnitudes x—vary continuously; one can then conceive the lines as divided into elements, within which the ratios of the magnitudes dx may be regarded as constant; and the problem then reduces to that of establishing for each point a general expression for the line-element ds ~~proceeding~~ emanating from that point—an expression, therefore, that will contain the magnitudes x and the magnitudes dx. In the second place, I assume that the length of the line-element remains unchanged, if we neglect magnitudes of the second order, when all of its points are subjected to the same infinitely small change of place—which implies at the same time that when all the magnitudes dx grow in the same ratio, the line-element is likewise altered in this ratio. Under these assumptions, the line-element will be able to be an arbitrary homogeneous first-degree function of the magnitudes dx, which remains unchanged when all the magnitudes dx change their sign, and in which the arbitrary constants are continuous functions of the magnitudes x. In order to find the simplest cases, I first seek an expression for the (n - 1)-tuply extended manifolds that lie everywhere equally distant from the origin-point of the line-element; i.e., I seek a continuous function of place, which distinguishes these manifolds from one another. This function will have either to decrease in all directions or to increase in all directions from that origin-point; I choose to assume that it increases in all directions, and therefore has a minimum in that point. If, then, its first and second differential quotients are finite, the first-order differential must vanish and the second-order one can never be negative; I assume that it is always positive. This differential expression of the second order then remains constant when ds remains constant, and grows in quadratic ratio when the magnitudes dx, and therefore also ds, change all in the same ratio; it is therefore equal to const$\cdot ds^2$, and consequently ds is equal to the square-root of an always positive entire homogeneous second-degree function of the magnitudes dx, in which the coefficients are continuous functions of the magnitudes x. For space, if one expresses the position of the points by rectangular coordinates, there results $ds = \sqrt{\sum (dx)^2}$; space is thus contained under this simplest case. The next most simple case would comprise the manifolds in which the line-element can be expressed by the fourth root of a differential expression of fourth degree. The investigation of this more general species would indeed demand no essentially different principles, but would be rather time-consuming and throw little new light upon the theory of space, especially since the results cannot be expressed geometrically; I therefore restrict myself to the manifolds where the line-element is expressed by the square-root of a differential expression of second degree. One can transform such an expression into another similar one by substituting for the n independent variable functions n new independent variables. But in this way one will not be able to transform each expression into each; for the expression contains $n\frac{n+1}{2}$ coefficients, which are arbitrary functions of the independent variables; and by introduction of new variables one will be able to satisfy only n relations and so to make only n of the coefficients equal to given magnitudes. The remaining $n\frac{n-1}{2}$ are then fully determined by the nature of the manifold to be represented, and thus $n\frac{n-1}{2}$ functions of place are required for the specification of the measure-relations of the manifold. The manifolds in which, as in the plane and in space, the line-element can be brought into the form $\sqrt{\sum (dx)^2}$ therefore constitute only a special case of the manifolds to be investigated here; they deserve a special name, and I shall therefore call these manifolds, in which the square of the line-element can be brought to a sum of squares of independent differentials, planar [or flat]. In order to be able now to survey the essential differences among all manifolds representable in the presupposed way, it is necessary to eliminate the differences that stem from the mode of representation; and this will be achieved by a choice of the variable magnitudes according to a definite principle.

2.

To this end conceive as constructed, from an arbitrary point, the system of all the shortest lines emanating from that point; the position of an indeterminate point will then be specifiable by the initial direction of the shortest line in which it lies, and by its distance along that line from the origin-point —it can therefore be expressed by the ratios of the magnitudes dx^0, i.e. the magnitudes dx at the origin of this shortest line, and by the length s of this line. Now introduce instead of dx^0 such linear expressions formed from them, $d\alpha$, that the initial value of the square of the line-element is equal to the sum of the squares of these expressions—so that the independent magnitudes are: the magnitude s and the ratios of the magnitudes $d\alpha$; and finally put instead of the $d\alpha$ such magnitudes $x_1, x_2, \ldots, x_n$, proportional to the former, that the sum of their squares is equal to s^2. If one introduces these magnitudes, then for infinitely small values of x the square of the line-element becomes equal to $\sum dx^2$; but the term of next order in the square of the line-element becomes equal to a homogeneous expression of the second degree in the $n\frac{n-1}{2}$ magnitudes $(x_1 dx_2 - x_2 dx_1)$, $(x_1 dx_3 - x_3 dx_1)$, ..., thus an infinitely small magnitude of the fourth dimension; so that one obtains a finite magnitude if one divides it by the square of the infinitely small triangle in whose vertices the values of the variables are $(0, 0, 0, \ldots)$, $(x_1, x_2, x_3, \ldots)$, $(dx_1, dx_2, dx_3, \ldots)$. This magnitude retains the same value, so long as the magnitudes x and dx are contained in the same binary linear forms [more clearly and correctly: so long as the ratios of the bilinear forms $(x_i dx_\iota - x_\iota dx_i)$ are preserved], or so long as the two shortest lines from the values 0 to the values x and from the values 0 to the values dx remain in the same surface-element; it thus depends only upon the place and the direction of that surface-element. It is obviously = 0 when the represented manifold is flat—i.e., when the square of the line-element is reducible to $\sum dx^2$; and it can therefore be regarded as the measure of the deviation of the manifold from flatness in this point and in this surface-direction. Multiplied by $-\frac{3}{4}$, it becomes equal to the magnitude which Herr Geheimer Hofrat Gauss has called the measure of curvature of a surface. We have previously found that for the specification of the measure-relations of an n-tuply extended manifold, representable in the form we have presupposed, $n\frac{n-1}{2}$ functions of place are necessary; when, therefore, the measure of curvature is given at each point in $n\frac{n-1}{2}$ surface-directions, the measure-relations of the manifold will be determinable therefrom—only supposing that no identical relations hold among these values (as in fact, speaking generally, they do not). The measure-relations of these manifolds, in which the line-element is repre-

[redacted]

sented by the square-root of a differential expression of second degree, can be thus expressed in a way entirely independent of the choice of the variable magnitudes. A quite similar way can be taken to this goal for those manifolds, too, in which the line-element is given by a less simple expression—e.g., by the fourth root of a differential expression of fourth degree. Then, generally speaking, the line-element could no longer be brought into the form of the square-root of a sum of squares of differential expressions; and so in the expression for the square of the line-element the deviation from flatness would be an infinitely small magnitude of the second dimension—whereas for the former manifolds it was an infinitely small magnitude of the fourth dimension. This peculiarity of those manifolds can therefore reasonably be called flatness in the smallest parts. But the most important peculiarity of these manifolds for the present purpose, on account of which they alone have been investigated here, is this, that the circumstances of the doubly extended ones can be represented by surfaces, and those of the multiply extended ones can be reduced to those of the surfaces they contain—a point that still requires a brief discussion.

3.

In the idea of surfaces there is always mixed, together with the inner measure-relations for which only the length of paths inside them comes into consideration, also their situation with respect to points that lie outside them. But one can abstract from the external circumstances, by subjecting the surfaces to such variations as leave unchanged the length of the lines within them: i.e., by conceiving them to be bent arbitrarily (without distention), and regarding all surfaces that arise from one another in this way as alike. Thus, e.g., all cylindrical or conical surfaces count as like a plane, because they can be formed from a plane by mere bending (in which the inner measure-relations remain the same, and all propositions about these measure-relations—hence all of planimetry—preserve their validity); by contrast, these same surfaces count as essentially different from the sphere, which cannot be transformed without distension into a plane. According to the preceding investigation, for a doubly extended magnitude wnose line-element can be expressed, as it can for surfaces, by the square-root of a differential expression of second degree, the inner measure-relations are characterized at each point by the measure of curvature. Now this magnitude admits in the case of surfaces the intuitive explication, that it is the product of the two [principal] curvatures of the surface at the point in question; or, again, that its product by an infinitely small triangle formed of shortest lines is equal to the excess of its angle-sum over two right angles, expressed in parts of the radius [i.e., in radian measure]. The first definition would presuppose the proposition that the product of the two [principal] radii of curvature remains unchanged on mere bending of the surface; the second, that at the same place the excess of the angle-sum of an infinitely small triangle over two right angles is proportional to its area. In order to give a palpable meaning to the measure of curvature of an _n_-tuply extended manifold at a given point in a given surface-direction through that point, one must start from the fact that a shortest line emanating from a point is fully determined when its initial direction is given. It follows that one will obtain a definite surface, if one extends to shortest lines all the initial directions emanating from the given point and lying in the given surface-element; and this surface has, at the given point, a definite measure of curvature, which is at the same time the measure of curvature of the _n_-tuply extended manifold at the given point and in the given surface-direction.

4.

There are still, before the application to space is made, a few considerations necessary concerning the flat manifolds in general—i.e., concerning those in which the square of the line-element is representable by a sum of squares of complete differentials.

In a flat _n_-tuply extended manifold the measure of curvature at each point in each direction is null; but according to the earlier investigation it is sufficient, in order to determine the measure-relations, to know that at each point that measure is null in $n\frac{n-1}{2}$ surface-directions whose measures of curvature are mutually independent. The manifolds whose measure of curvature is everywhere = 0 can be considered as a special case of those manifolds whose measure of curvature is everywhere constant. The common character of these manifolds, whose measure of curvature is constant, can also be expressed thus: that the figures within them can be moved without distension. For obviously the figures within them could not be arbitrarily displaceable and rotatable unless the measure of curvature were the same at every point in all directions. But conversely the measure-relations of the manifold are completely determined by the measure of curvature; therefore the measure-relations about one point in all directions are exactly the same as about another, and thus the same constructions can be carried out from it, and consequently in the manifolds of constant curvature figures can be given every arbitrary position. The measure-relations of these manifolds depend only upon the value of the measure of curvature, and in respect of the analytical representation it may be remarked that, if one denotes this value by α, the expression for the line-element can be given the form

$$\frac{1}{1 + \frac{\alpha}{4}\sum x^2}\sqrt{\sum dx^2}\ .$$

5.

For geometrical elucidation, the consideration of the <u>surfaces</u> with constant measure of curvature can serve. It is easy to see that the surfaces whose measure of curvature is positive can always be wrapped upon a sphere whose radius is equal to 1 divided by the [square-]root of the measure of curvature; but to survey the entire manifold of these surfaces, let us give one of them the shape of a sphere, and the others the shape of surfaces of rotation tangent to that sphere at the equator. The surfaces with greater measure of curvature than this sphere will then be tangent to the sphere from inside, and will assume a shape like the outer part of the surface of a ring—the part facing away from the axis; they would admit of being wrapped upon zones of spheres of smaller radius, but going more than once around. The surfaces with smaller positive measure of curvature will be obtained, if from spherical surfaces of larger radius one cuts out a piece bounded by two great circles, and fuses together the cut-lines. The surface with the measure of curvature null will be a cylindrical surface over the equator; and the surfaces with negative measure of curvature will be tangent externally to this cylinder, and will be formed like the inner part of the surface of a ring—the part facing towards the axis. If one thinks of these surfaces as locus for pieces of surface that are movable within them—as space is locus for bodies—then in all these surfaces the pieces of surface are movable without distension. The surfaces with positive measure of curvature can always be so formed that the pieces of surface can also be moved arbitrarily without bending—namely, formed into spherical surfaces—but those with negative measure of curvature cannot. Besides this independence of the piece of surface from its place, there also holds for the surface with the measure of curvature null an independence of direction from place, which does not hold for the other surfaces.

III. APPLICATION TO SPACE.

1.

After these investigations concerning the determination of the measure-relations of an <u>n</u>-tuply extended magnitude, the conditions can now be given which are

Euc. geom.

sufficient and necessary to the determination of the measure-relations of space, if independence of the [length of] lines from position and representability of the line-element by the square-root of a differential expression of second degree —in short, flatness in the smallest parts—is presupposed.

They can, first, be expressed thus: that the measure of curvature at each point in three surface-directions is = 0; and therefore the measure-relations of space are determined, if the angle-sum in triangles is everywhere equal to two right angles.

…ut Helmholtz demands

But, second, if one presupposes, as Euclid does, an existence independent of position not merely for lines, but also for bodies, it follows that the measure of curvature is everywhere constant; and the angle-sum is then determined in all triangles when it is determined in one.

∃ a coherent …tion of distant parallelism' …ransport of vectors

Third and finally, instead of assuming the length of lines as independent from place and direction, one could also presuppose an independence of their length and direction from place. According to this conception, the changes of place or differences of place are complex magnitudes, expressible in three independent units.

not imaginary, but vectorial

→ you only have affine structure in Euclidean space

2.

In the course of the considerations so far, first the extension-relations or domain-relations were separated from the measure-relations, and it was found that for the same extension-relations various measure-relations are conceivable; then the systems of simple measure-determinations were sought, by which the measure-relations of space are fully determined and of which all propositions about those relations are a necessary consequence; it remains now to discuss the question, how—in what degree, and within what range—these presuppositions are warranted by experience. In this respect there is an essential difference between the mere extension-relations and the measure-relations: in that for the former, where the possible cases constitute a discrete manifold, the pronouncements of experience are indeed never completely certain, but are not inexact; whereas for the latter, where the possible cases constitute a continuous manifold, every determination from experience remains always inexact—no matter how great may be the probability that that determination is approximately correct. This circumstance becomes important for the extension of these empirical determinations beyond the limits of observation into the immeasurably large and immeasurably small; for beyond the limits of observation, the second sort may obviously become ever more inexact—but the first sort cannot.

…f. Peirce, …e Doctrine of …cessity Examined

For the extension of space-constructions into the immeasurably great, unboundedness and infinitude are to be distinguished; the former pertains to the extension-relations, the latter to the measure-relations. That space is an unbounded triply extended manifold is a presupposition which is applied in every conception of the external world; a presupposition in accordance with which the domain of actual perceptions is each moment supplemented, and the possible places of a sought-for object are constructed; and a presupposition which in these applications is continually confirmed. The unboundedness of space therefore possesses a greater empirical certainty than any external experience. But from this its infinitude in no way follows; on the contrary, if one presupposes independence of bodies from place, and thus ascribes to space a constant measure of curvature, it would necessarily be finite as soon as this measure of curvature had a positive value, no matter how small. One would obtain, on prolonging to shortest lines the initial directions that lie in a surface-element, an unlimited surface with constant positive measure of curvature--thus a surface which in a flat triply extended manifold would assume the shape of a spherical surface, and which consequently is finite.

3.

The questions about the immeasurably great are, for the explanation of nature, idle questions. But it is otherwise with the questions about the immeasurably small. Upon the exactness with which we pursue the phenomena into the infinitely small essentially rests our knowledge of their causal connection. The advances of the last century in the knowledge of mechanical nature are almost solely conditioned by the exactness of construction which has become possible through the invention of the infinitesimal analysis and through the simple principles [Grundbegriffe], discovered by Archimedes, Galilei, and Newton, which present-day physics has at its disposal. In those natural sciences, however, where the simple principles for such constructions are still wanting, one pursues the phenomena, in order to know the causal connection, as far into the spatially small as the microscope will allow. The questions about the measure-relations of space in the immeasurably small therefore do not belong to the idle ones.

If one presupposes that bodies exist independently of place, then the measure of curvature is everywhere constant; and it follows from the astronomical measurements that that measure cannot be different from zero--at any rate its reciprocal would have to be equal to an area in relation to which the region accessible to our telescopes is vanishingly small. But if such an independence of bodies from place does not obtain, then one cannot infer the measure-relations in the infinitely small from those in the large; in that case the measure of curvature can have an arbitrary value at each point in three directions, if only the total curvature of every measurable part of space does not differ noticeably from zero; still more complicated relationships can occur, if the presupposed representability of a line-element by the square-root of a differential expression of second degree does not obtain. Now the empirical concepts in which the spatial measure-determinations are grounded—the concept of the solid body and of the light-ray—appear to lose their validity in the infinitely small; it is therefore very well conceivable that the measure-relations of space in the infinitely small are not in accord with the presuppositions of geometry—and one would in fact have to adopt this assumption, as soon as the phenomena were found to admit of simpler explanation by this means.

The question of the validity of the presuppositions of geometry in the infinitely small is bound up with the question of the inner ground of the measure-relations of space. In this question, which may well be still reckoned to the account of the theory of space, the earlier remark comes to application, that for a discrete manifold the principle of the measure-relations is already contained in the concept of this manifold, but for a continuous one must come from somewhere else. Therefore either the reality that lies at the basis of space must constitute a discrete manifold, or the ground of the measure-relations must be sought outside of it, in binding forces that act upon it.

The decision of these questions can only be found by proceeding from the traditional and empirically confirmed conception of the phenomena, of which Newton has laid the foundation, and gradually revising this, driven by facts that do not admit of explanation by it; such investigations as, like that conducted here, proceed from general concepts, can only serve to ensure that this work shall not be hindered by a narrowness of conceptions, and that progress in the knowledge of the connections of things shall not be hampered by traditional prejudices.

ry unusual for-seeing ... of ...tion of th to physics - h does not ... job of ... ~~...~~ what ... a priori simply ... of explo... ...ilities

, contra Poincaré, wh ... that we should hold onto certain ...ptions because of habit -)

This leads over into the domain of another science, into the domain of physics, upon which the nature of the present occasion makes it inappropriate to enter.

SYNOPSIS

Plan of the investigation.

I. Concept of an _n_-tuply extended magnitude[1]).

§ 1. Continuous and discrete manifolds. Definite parts of a manifold are called quanta. Division of the theory of continuous magnitudes into the theory

1. of the mere domain-relations, in which an independence of the magnitudes from place is not presupposed,
2. of the measure-relations, in which such an independence must be presupposed.

§ 2. Generation of the concept of a simply, doubly,..., _n_-tuply extended manifold.

§ 3. Reduction of the specification of place in a given manifold to specifications of quantity. Essential criterion of an _n_-tuply extended manifold.

II. Measure-relations of which a manifold of _n_ dimensions is susceptible[2]), under the presupposition that the lines possess a length independently of position, therefore that each line is measurable by each.

§ 1. Expression for the line-element. Those manifolds will be considered as flat, in which the line-element is expressible as the root of a sum of squares of complete differentials.

§ 2. Investigation of the _n_-tuply extended manifolds in which the line-element can be represented by the square-root of a differential expression of second degree. Measure of their deviation from flatness (measure of curvature) at a given point in a given surface-direction. For the specification of their measure-relations it is (under certain restrictions) admissible and sufficient that the measure of curvature be given arbitrarily at each point in $n\frac{n-1}{2}$ surface-directions.

§ 3. Geometric elucidation.

§ 4. The flat manifolds (in which the measure of curvature is everywhere = 0) can be regarded as a special case of the manifolds with constant measure of curvature. These latter can also be defined by the fact that in them there is independence of the _n_-tuply extended magnitudes from place (mobility of those magnitudes without distension).

§ 5. Surfaces with constant measure of curvature.

---

[1]) Art. I constitutes at the same time the preliminary work for contributions to Analysis situs.

[2]) The investigation of the possible measure-determinations of an _n_-tuply extended manifold is very incomplete, although for the present purpose perhaps sufficient.

III. Application to space.

§1. Systems of facts which suffice to determine the measure-relations of space, as geometry presupposes these.

§2. To what extent is the validity of these empirical determinations probable beyond the limits of observation in the immeasurably large?

§3. To what extent in the immeasurably small? Connection of this question with the explanation of nature[1]).

---

[1]) The §3 of Art. III. still requires revision and further elaboration.