# CHRISTIAAN HUYGENS
# ON THE MOTION OF BODIES
# RESULTING FROM IMPACT



## Hypotheses

1. Any body once moved continues to move, if nothing prevents it, at the same constant speed and along a straight line.

2. Whatever be the cause of the rebound of hard bodies from mutual contact when they collide with one another, we posit that when two equal bodies with equal speed collide directly with one another from opposite directions each rebounds with the same speed with which it approached.

   They are said, moreover, to collide directly when both the motion and the contact occur on the straight line joining their centers of gravity.

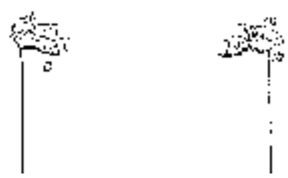
3. The motion of bodies and their equal and unequal speeds are to be understood respectively, in relation to other bodies which are considered as at rest, even though perhaps both the former and the latter are involved in another common motion. And accordingly, when two bodies collide with one another, even if both together are further subject to another uniform motion, they will move each other with respect to a body that is carried by the same common motion no differently than if this motion extraneous to all were absent.

   Thus, if someone conveyed on a boat that is moving with a uniform motion were to cause equal balls to strike one another at equal speeds with respect to himself and the parts of the boat, we say that both should rebound also at equal speeds with respect to the same passenger, just as would clearly happen if he were to cause the same balls to collide at equal speeds in a boat at rest or while standing on the ground.

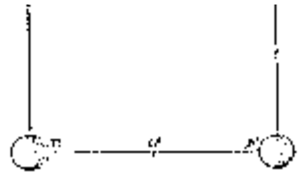
   Having posited these things about the collision of equal bodies, we will demonstrate by what laws they are moved by each other; we will bring in in their [proper] place other hypotheses which we will need for cases of unequal bodies.

## Proposition I

If a body at rest is struck by another equal body, after contact the latter will be at rest, and the body at rest will acquire the speed that was in the striking body.



Imagine a boat to be carried along the shore by a river, close enough to the shore that a passenger standing in it can reach out his hands to a friend on the shore. Let the passenger hold in his hands A and B two equal bodies E, F, suspended on strings, the distance EF between them being divided in

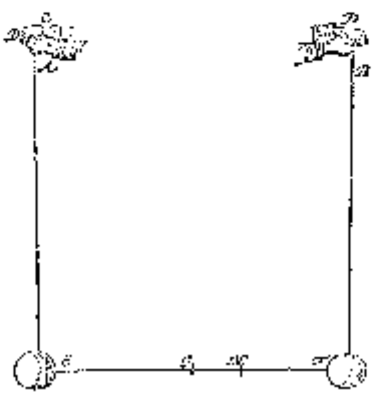


two by point G. And moving his hands with an equal motion to a mutual meeting, with respect to himself and the boat, he will also cause the balls E, F to collide with each other at equal speeds; and so it is necessary that they will rebound from mutual contact also at equal speeds with respect to the same passenger and boat. Meanwhile, assume that the boat is carried toward the left at speed GE, the same speed namely with which the left hand A has been carried toward the right.

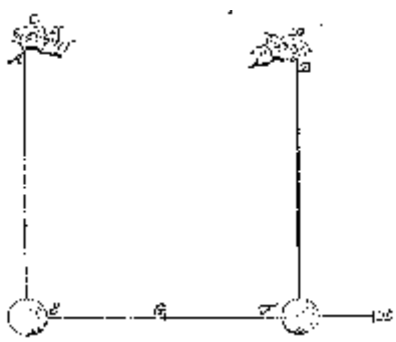
It is thus clear that hand A has remained unmoved with respect W to the shore and the friend standing on it and hand B has been moved with respect to the same friend at speed FE, the double of this GE, or FG. Wherefore, if the friend standing on the shore is assumed to grasp with his hand C the hand A of the passenger, and with it the end of the string supporting ball A, and he grasps with hand D the hand B of the passenger, which supports the string from which F hangs, it appears that, while the passenger causes the balls E, F to meet at equal speeds, at the same time the friend standing on the shore has struck ball E at rest with ball F moved at speed FE with respect to the shore and himself. Moreover, it is evident that the passenger moving his balls as described is not hindered therein by the fact that the friend standing on the shore grasps his hands and the ends of the strings, since he only accompanies their motion and adds no impediment to it. By the same reasoning, neither is the friend standing on the shore and carrying ball F against the immobile ball E hindered therein by the fact that the passenger has his hands joined with his, inasmuch as hands A and C are both at rest with respect to the shore and the friend, while the two D and B are moved at the same speed FE. But, because, as has been said, balls E, F, after mutual contact, rebound at equal speeds with respect to the passenger and the boat, i.e. ball E at speed GE and ball F at speed GF, and the boat itself proceeds in the meantime toward the left at speed GE, or FG, it follows, with respect to the shore and the friend standing on it, that ball F will remain at rest after the impulse and the other E, with respect to the same, will proceed toward the left at twice the speed GE, that is, at the speed FE at which ball F pushed toward E. And so we have shown that, on motionless ground, when an unmoved body is struck by an equal body, the latter loses all motion after contact, while the former acquires all. Q.E.D.

# Proposition II

If two equal bodies moved at unequal speeds strike each other, they are moved with exchanged speeds after contact.



Let body E be moved toward the right at speed EH, and let F, equal to it, tend first from the opposite direction at the smaller speed FH; they will meet therefore at H. I say that after their mutual collision body E will be moved toward the left at speed FH, and F toward the right at speed EH.



For imagine a man standing on the shore to effect the said motions by supporting with his hands C, D the ends of strings from which these [bodies] are suspended and causing his hands, and at the same time bodies E and F, to come together at the said speeds EH, FH. Let the distance EF again be bisected at G, and imagine a boat to pass by toward the right at speed GH, in which [boat] another man is standing, with respect to whom ball E will be moved at speed EG only, and ball F at speed FG, so that with respect to him the two balls are moved at equal speeds toward a mutual collision. Whence, if he is assumed to grasp with his hands A, B the hands C, D of the friend standing on the shore, and with them the ends of the strings on which the balls are suspended, the result will be that, at the same time, the one standing on the shore will cause them to come together at speeds EH, FH and the one conveyed by the boat will cause them to come together at the equal speeds EG, FG. And so it is evident that, with respect to the latter, both will rebound from contact also at equal speeds, i.e. E at speed GE and F at speed GF. But the boat continues in the meantime to move at speed GH. And so, with respect to the shore and the man standing on it, F will have a speed composed of both GF and GH, i.e. equal to EH, and E the speed HF by which the speeds GE, GH differ from one another. And thus we have shown that if a man standing on the shore strikes the balls E and F against each other at speeds EH, FH, after the impulse E will rebound at speed FH, and F at speed EH; Q.E.D.

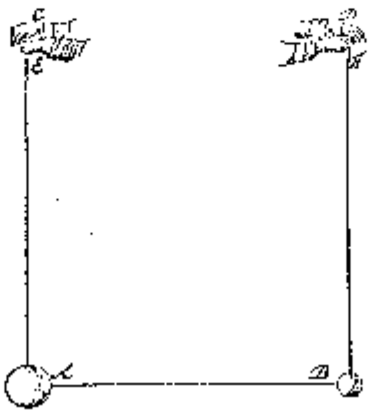
Now let each body E and F be moved toward the right, E at speed EH, the preceding F at the smaller speed FH; E will, therefore, overtake F and they will meet at H. I say again that, after contact, F will proceed at speed EH, and E will follow after at speed FH. The demonstration is the same as above.

# Hypothesis 4

If a larger body meets a smaller one at rest, it will give it some of its motion and hence lose something of its own.

# Proposition III

A body however large is moved by impact by a body however small and [moving] at any speed.



Imagine a boat carried along the shore of a river. A passenger standing in the boat supports bodies A and B hung on strings; let A, which he holds in his left hand, be greater and B less. Let him hold his right hand D, which supports body B, unmoved with respect to himself and the boat, and toward it let him move hand C, together with body A, at any speed AB. Hence, B is impelled and body A loses something of its speed and hence continues to move toward the right at a speed less than AB. But, while these things are happening, assume the boat to be carried toward the left at speed BA, and, while the passenger moves body A at speed AB with respect to himself and the boat, the same body remains unmoved with respect to the shore and a spectator standing on it, and similarly hand C [remains unmoved]. The other hand D with body B will be moved, with respect to the same spectator, toward the left at speed BA, since we have assumed it unmoved with respect to the boat and the boat is moved toward the left at speed BA. Whence, if the spectator standing on the shore is assumed to grasp with his hands, E, F the hands C, D of the passenger, it is clear that, while the latter moves ball A toward B, which is unmoved with respect to himself, at the same time the former will move B toward A, which rests unmoved with respect to him and the shore. But we have said that by the impulse ball A, with respect to the passenger and the boat, will be carried toward the right at a velocity less than AB; yet the boat is carried toward the left at speed BA. Hence, with respect to the shore and the spectator, standing on it, it is manifest that A will be moved somewhat toward the left by the impulse. And so it has been shown that on the motionless ground when a body A, at rest and however larger, is struck by a body B, however small and [moving] at any speed, body A will be moved; Q.E.D.

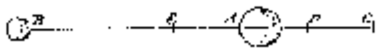
# Hypothesis 5

When two hard bodies meet each other, if, after impulse, one of them happens to conserve all the motion that it had, then likewise nothing will be taken from or added to the motion of the other.

# Proposition IV

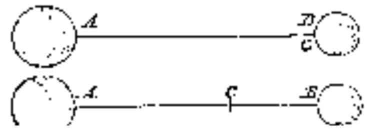
Whenever two bodies collide with one another, the speed of separation is the same, with respect to each other, as that of approach.

This is evident for equal bodies from Proposition II. Therefore, let them now be unequal, and take first the case in which a larger A at rest is struck by a smaller body B moving to the right at speed BA. I say that after contact they will be separated at the same speed BA, such that, if in one part of time body B traverses space BA, after another similar part of time they will again be separated by a space equal to this BA.

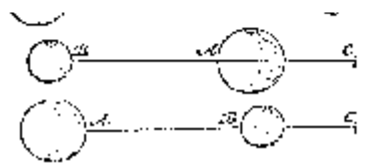


For it holds that A will receive some speed by the impulse of body B; let this be AC, which will necessarily be less than the speed BA with which B was moved. Now, if A were equal to this B, then it would receive precisely speed BA by the impulse. Let AC be divided into two equal parts by point D, and let AE be equal to AD. If, therefore, we were to imagine these motions taking place in a boat that was proceeding toward the left at speed DA, it is necessary that, before the impulse, body A, which was at rest in the boat, would have been moving toward the left at the said speed DA with respect to the shore. And, after the impulse, when it is said to be moved in the boat toward the right at speed AC, and this boat is carried in the opposite direction at speed DA, A will be moved, with respect to the shore, at speed DC, of AD, to the right. And so, with respect to the shore, body A maintains the same speed before and after impulse. Whence B, with respect to the same, likewise necessarily loses nothing of its speed. But, before collision, B was moved at speed BE to the right with respect to the shore, because in the boat it had speed BA toward the right and this boat had speed DA, or AE, in the opposite direction. Therefore, after collision too it will have to be moved at speed BE with respect to the shore, but toward the left, unless the slower motion of body A toward the right can hinder it. Therefore, since after impulse B is moved, with respect to the shore, at speed EB toward the left, and A toward the right at speed AD, or EA, it is necessary that they separate from each other at the speed composed of both BE, EA, i.e. at speed BA, and that with respect not only to the shore but also the boat, since they separate at that speed in reality. For what happens to things colliding in a moving boat must also happen in the same manner everywhere outside the boat.

With this case demonstrated, the rest follows easily; and four different [cases] remain: either the smaller body is at rest, or both move with opposite motions, or the smaller follows the larger with a faster motion, or conversely; these may all be set forth together.



For let body A as before be greater than B, and let A be moved at speed AC, while B either is altogether at rest or has speed BC. Since, therefore, the bodies so moved have the speed AB with

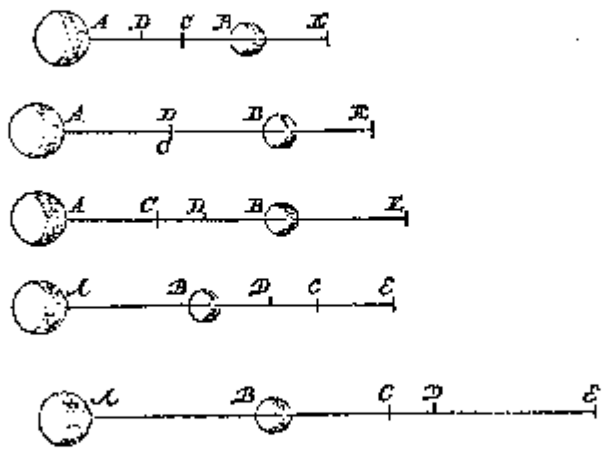


respect to each other, I say that they will also be separated after impulse at the same speed.

For if these motions are again thought to take place in a boat which is proceeding at speed CA, that is, the same speed at which body A is moved, but in the opposite direction, it is clear that, with respect to the shore, A will remain unmoved and B, in each case, will meet it at speed BA. But A is greater than B; hence, the preceding case obtains, from which it is clear that after impulse the bodies should separate at the same speed AB with respect to the shore. Whence it is most evident that they will recede from one another at this speed also with respect to the boat and in fact [*revera*].

# Proposition V

If two bodies each collide again at the speed at which they rebounded from impulse, after the second impulse each will acquire the same speed at which it was moved toward the first collision.



Assume A to have been moved at speed AC, B at speed BC, and the two of them to have collided with one another such that A rebounded at speed CD and B at speed CE. Afterwards, let both return to a collision at these same speeds, i.e. A at speed DC, B at speed EC. I say then that A will rebound at speed CA, B at speed CB, at which speeds they first tended toward collision. For if, while they move toward a second collision, A at speed DC, B at speed EC, we imagine these motions to happen in a boat, which is proceeding at speed AD, then, with respect to the shore, A will be moved at speed AC, because it is moved in the boat at speed DC and the boat is moving at speed AD; and B [will be moved] with respect to the shore at speed BC. Now, because DE is equal to AB, having taken away the common [segment] DB, BE will be equal to AD; therefore, the boat is moved at speed BE, and B in the boat at speed EC, whence, with respect to the shore, B will be moved at speed BC, as we have said. It is necessary, therefore, with respect to the same shore, that they separate after collision, A at speed CD and B at speed CE; for it was assumed at the beginning that, if A tends toward collision at speed AC, and B at speed BC, they will separate after impulse, A at speed CD and B at speed CE. Hence, while A is moved with respect to the shore at speed CD, and the boat at speed AD, it will result that A is moved in the boat at speed CA. Also, since B is moved with respect to the shore at speed CE, and the boat at speed AD or BE, the speed of this B in the boat will be CB. Therefore, what in the boat are moved toward collision at speeds DC, BC will, it has been shown, rebound in the boat at speeds CA, CB, whence the same thing necessarily happens anywhere, and the proposition holds.

# Proposition VI

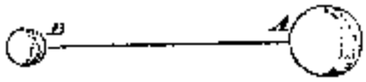
When two bodies collide with one another, the same quantity of motion in both taken together does not always remain after impulse what it was before, but can be either increased or decreased.

The quantity of motion is so measured [*aestimatur*] that in unequal bodies moved equally fast each constitutes a greater quantity of motion by as much as it is greater; but in equal bodies moved at unequal speeds, by as much as one is faster than the other. Let us therefore demonstrate what has been proposed.

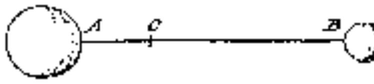
Let body A be greater than B; let A be at rest and B be moved toward it at speed BA; therefore, A will be moved and will acquire some speed, say AC. B, however, will rebound at speed AD, such that the whole speed CD which they will have with respect to one another is equal to speed AB. Wherefore, if body A is equal to this B, the same quantity of motion will obtain after the impulse as before, for clearly it constitutes the same quantity of motion] whether two bodies equal to this B are moved, the one at speed AD, the other at speed AC, or B alone is moved at speed CD or BA. Body A, however, is greater than B; hence it is clear that a greater quantity of motion is constituted, when after impulse body A is moved at speed AC and body B at speed AD, than beforehand, when B alone had speed BA. That the quantity of motion can also be diminished is shown thus: for if, when B strikes body A at rest with speed BA, speed AC is acquired by this A and speed AD remains in B, it will in turn be the case that, if A approaches at speed CA, and B from the opposite direction at speed DA, after contact A will remain devoid of motion, while B rebounds at speed AB. Whence, from what has been shown beforehand, the quantity of motion will then be less after collision than it was before.

# Proposition VII

If a greater body strike a smaller one at rest, it gives it a velocity less than double its own.




Let body A strike at speed AB the smaller B at rest. I say that a speed less than double AB will be impressed in this B. For, because after impulse the bodies should separate from each other at the same speed AB, it would be necessary that, if the speed of body B were double the speed AB, A after impulse would follow body B at the same speed AB, which cannot happen. And [if the speed of B were] more than double, it would be necessary for A after impulse to continue to move at a speed greater than AB, which is similarly absurd. Whence the proposition holds.



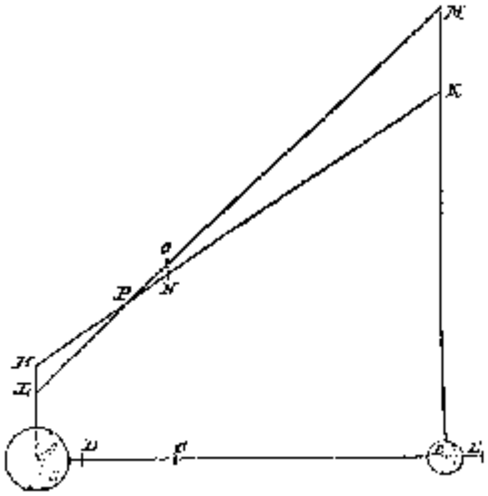

Just as it has been universally demonstrated for equal bodies how [*qua ratione*] one transfers motion to the other, on the assumption that equals colliding with one another at equal speed rebound equally, so, for bodies of diverse magnitude, all cases can be determined, of which many exist, on the following supposition: if two unequal bodies are moved to mutual collision, and the speeds correspond inversely to the magnitudes, then each will be moved back from contact at the same speed at which it approached.

For example, if A is three times B, and the speed BC, at which B is moved, is three times the speed AC, at which A is moved, then, the collision having occurred at C, each body will rebound at the same speed at which it was moved formerly. Further, because this is not equally as evident as that which has been assumed concerning equal bodies (even though it is not alien to reason and agrees above all with experiments), we will try to confirm it by demonstration.

It certainly holds that, whenever two heavy bodies are moved downward by naturally accelerated motion, the ratio of the spaces traversed by them is the duplicate of the ratio of the highest degrees of speed acquired by them. For this has been demonstrated by Galileo in the third dialogue on motion and observed through innumerable and most exquisite experiments, just as this also [has been shown], that the speed acquired by a falling body can restore it to the same height from which it descended. The demonstrations of both of these are also set forth in what I have written concerning the clock. And from these the said theorem can be demonstrated.

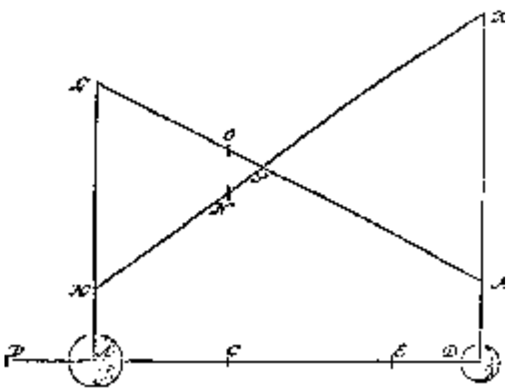
# Proposition VIII

If two bodies, the speeds of which correspond inversely to the magnitudes, collide with each other from opposite directions, each will rebound at the same speed at which it approached.

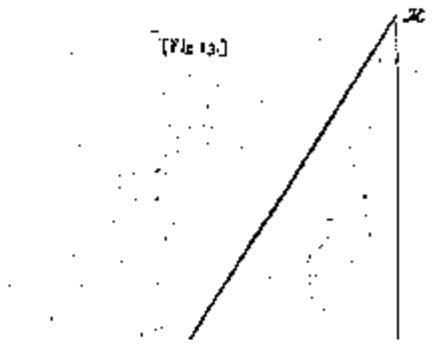


Let bodies A and B, of which the former is greater than the latter, collide with each other and let the speed of body B, which is BC, have to the speed of body A, which is AC, the same ratio as that which the magnitude of A has to the magnitude of B. It is to be shown that, after mutual contact, each will rebound at the same speed at which it approached, to wit, A at speed CA and B at speed CB. It holds, however, that if A is reflected at speed CA then B will be reflected at speed CB; otherwise, the speed of separation with respect to each other would not be the same as that of approach. If, therefore, body A does not rebound at speed CA, let it first rebound, if it can be done, at a smaller speed CD; hence, B will rebound at a speed CE greater than that at which it approached, such that DE is equal to AB. Let us suppose body A to have acquired the prior speed AC, at which it tended toward collision, by falling from the height HA, such that after it has fallen to A it changes the perpendicular motion to a horizontal [motion], the speed of which is AC; [suppose] moreover body B to have similarly acquired the speed BC by falling from height KB. These heights are, therefore, in the duplicate ratio of the speeds, that is, as the square of AC is to the square of CB so HA is to KB. But then if after impulse bodies A and B convert their horizontal motions, the speeds of which are measured by CD, CE, into perpendicular motions upward, it holds that body A will arrive at height AL such that as AL is to AH so the square of CD is to the square of CA. For when AL has this sort of ratio to AH, it is certain that in falling from height LA the body will acquire velocity CD; whence, conversely, having velocity CD, it will be able to attain height AL, according to what was posited above. And body B, by converting speed CE into perpendicular motion upward, will arrive at height BM, such that MB is to KB as the square of CE is to the square of CB. Draw connecting lines HK, LM, which necessarily will intersect one another, say at P, and divide both similarly at N and 0, such that as magnitude B is to [magnitude] A, i.e. as AC is to CB, so HN is to NK, and LO to OM. Since, then, the center of gravity of body A is located at H and the center of gravity of body B at K, the center of gravity composed of these is at point N. And after they fall from H and K and, following mutual collision, then lift themselves up to L and M, the center of gravity composed of them will be at O; which cannot happen, since, as we shall soon show, point O is higher than N. For in mechanics it is a most certain axiom that the common center of gravity of bodies cannot be raised by a motion that arises from their weight. Moreover, that point O is higher than N is shown thus: the excess of the square of EC over the square of BC is equal to the two rectangles CBE together with the square of BE, that is, to the rectangle that results from BE and the two [lines] EC, CB [taken] as one.

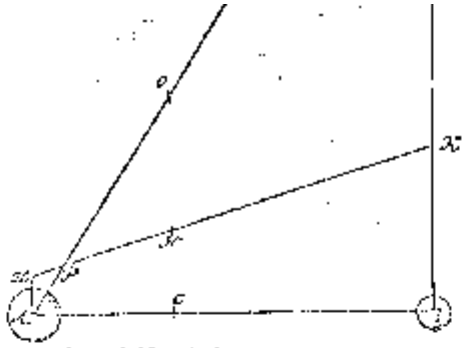
Similarly, the excess of the square of AC over the square of CD is equal to the rectangle formed by AD and the two [lines] AC and CD [taken] as one. But AD is equal to BE, since AB is equal to DE. And thus it is clear that the first excess, namely of the squares of EC, CB, is to the latter excess of the squares of AC, CD as both EC, CB taken together to both AC, CD taken together. But, since the two EC and CB are greater than twice CB, and the two AC, CD taken together less than twice AC, the ratio then of the two together EC, CB to AC, CD both together will be greater than [that] of CB to CA; and therefore the excess of the square of EC over the square of CB to the excess of the square of AC over the square of CD will have a ratio greater than that of BC to CA. And because the square of EC is to tile square of CB as MB is to BK in length, by division the excess of square EC over square CB will be to the square of CB as MK to KB. Moreover, the square of CB is to the square of CA as line KB is to HA. And, square CA is to its excess over square CD as HA to HL, for square AC was to square CD as HA to AL. And so, by equals, the excess of square EC over square CB to the excess of square AC over square CD will be as MK to HL. Whence the ratio of MK to HL will also be greater than [that of] BC to CA. But MK is to HL as MP to PL, and BC to CA as MO to OL. Therefore, the ratio of MP to PL will also be greater than [that of] MO to OL; and, by composition, the ratio of ML to IE' will be greater than [that of] ML to LO. And so LO will be greater than LP, whence it is clear that point O will fall on that side of the intersection P that is toward M; moreover, the [line] joining points O [and] N will be parallel to the perpendiculars MB, HA, since the lines LM, HK are divided according to the same ratio by these points. Therefore, just as M is higher than K, so O clearly is also higher than N, which remained to be demonstrated.



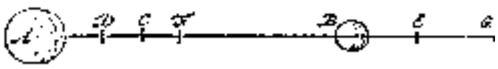
Now, if it can be done, let body A be reflected from collision at a speed CD greater than the CA at which it moved toward collision. CD will then be less than CB, which was the speed of body B before collision. For, if B were not less than A, but equal to it, then A would rebound by impulse at speed CB, and B would be reflected from collision at speed CE, such that DE were equal to AB. Now imagine the other things to have been done and the construction to have been carried out as in the preceding case; it will therefore turn out that L is higher than H, since DC is greater than AC, and that M is lower than K, since EC is less than CB. It will be shown again as before that the difference of squares DC, CA is to the difference of squares BC, CE as the two [lines] AC, CD taken together to the two [lines] EC, CB. And since the latter are together less than twice CB, the former greater than twice AC, the ratio of the two AC, CD together to the two EC, CB will be greater than [that of] AC to CB. And so the difference of the squares l)C, DA to the difference of the squares BC, CE will have a greater ratio than AC to CB. But it will be demonstrated further that the said difference to the said difference is as LH to KM. Hence, the ratio LH to KM, i.e. LP to PM, is also greater than AC to CB, i.e. than LO to OM; whence point O will fall on that side of the intersection P which is toward L. Moreover, as before, ON is parallel to LH. Hence, just as point L is higher than H, so O will be higher than N; but this, for the same reason as we gave in the preceding case, is absurd.



Now, if it be said that body A remains at rest after collision and only B is reflected, then it would be reflected at speed AB, since the bodies before collision had speed AB with respect to each other. Assuming, then, as before, the speed BC to have been acquired by body B in falling from height KB, it follows that, if one so arranges it that square CB is to square AB as BK to BM in length, this BM will be the height to which body B could ascend if it converted its



horizontal motion, by which it is moved at speed AB, into perpendicular motion upward. But since, after collision, body A is said to be devoid of motion, it will remain on line AB. And so, if the connecting line MA is drawn and is cut at O such that AO to OM is as AC to CB, O will be the point to the height of which the center of gravity of both bodies will ascend. But, if the bodies are placed at H and K, from where they are assumed to fall, their common center of gravity is at point N, which similarly divides line HK according to the ratio AC to CB. Thus, if it be further shown that point O is higher than point N, the demonstration will be reduced to the same absurd [result] as above. And this is shown thus: Since square AB is to square BC as MB is to BK in length, by division the excess of square AB over square BC will be to square BC as MK to KB. Moreover, as square BC is to square CA so is KB to HA, and this has been posited, just as in the first case. Therefore, by equals, the excess of square AB over square BC will be to square CA as to HA. But the ratio of the said excess to square CA is altogether greater than [that of] line BC to CA; thus MK to HA, i.e. MP to PA, will also be a greater ratio than BC to CA, i.e. than MO to OA. And therefore, by composition, MA to AP will be a greater ratio than MA to AO, whence it is clear that point O falls on the side of intersection P that is toward M. But M is higher than K; hence, since ON is necessarily parallel to this MK, point O will also be higher than N, which remained to be shown.



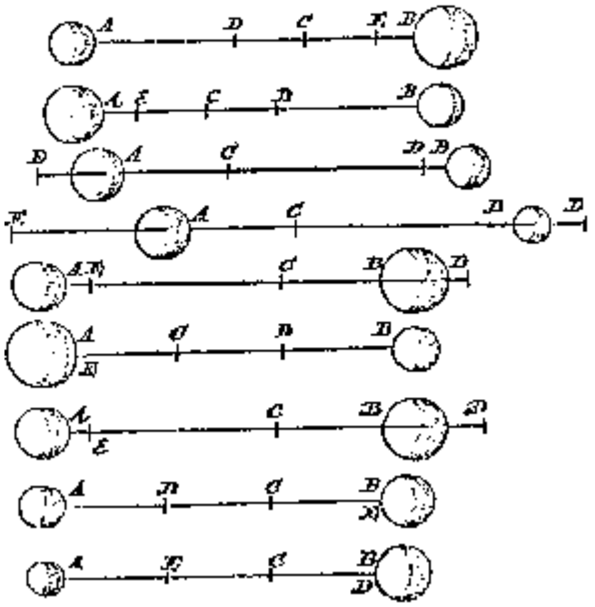
If, finally, body A is said after collision to continue to move in the same direction at speed CF, [the speed] will certainly not be greater than AC, at which it was moved before collision; moreover, body B will have to run before it at speed CG, the excess of which over speed CF, i.e. FG, is equal to AB. That this cannot take place is confirmed thus: take CD equal to CF, and then DE equal to AB. Hence, CE is made as much less than ED as CG is greater than this ED, of FG. But, since by assuming, as in the first case, that body A has been turned back from collision at speed CD, it would turn out that speed CE could not fit body B, lest the absurd [result] come about that, the motions along the horizontal being converted into perpendicular motions, the composite weight of the bodies would ascend higher than whence it descended. The same is much more necessary if body B acquires a speed CG still greater than CE, and A were to have a speed CF equal to this CD. Therefore body A does not continue to move in the same direction after collision. Wherefore it remains that it is moved backward at a speed CA as great as that at which it formerly tended toward collision, and thus B will rebound at speed CB. Q.E.D.

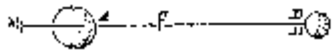
# Proposition IX

Given two unequal bodies colliding directly, each of which is moving, or just one, and given the speed of both, or of the one if the other is at rest, to find the speeds at which both are moved after collision.

Let body A be moved toward the right at speed AD; let B either be moved in the opposite direction, or run ahead in the same direction, at speed BD, or be at rest, i.e. such that point D falls on B. Therefore the speed with respect to each other will be AB.

Divide AB at C such that AC is to CB as B to A in magnitude, and take CE equal to CD. I say that EA will be the speed of body A after collision, EB that of body B; and this in the direction determined by the order of the points EA, EB. But, if point E falls on A, body A will be brought to rest; if E falls on B, body B will be at rest.





For, if we were to demonstrate that these things hold in a boat that is conveyed at a uniform speed, it would also hold that they would turn out the same way on motionless ground. Thus, imagine a boat to be carried along the shore of a river and a passenger standing in the boat to support with a his hands F, G balls A, B, suspended on strings, which by so moving at speeds AD and BD with respect to himself and the boat, he causes to collide at point D; moreover, assume the boat to proceed at speed DC in the direction indicated by the order of the points D, C. It will therefore result that ball A is moved, with respect to the shore and a spectator standing on it, at speed AC toward the right, because with respect to the boat it had speed AD. Moreover, ball B, since in the boat it has speed BD, will have with respect to the shore speed BC toward the left. And therefore, if the spectator standing on the shore grasps with his hands H, K the hands F, G of the passenger, and with them the ends of the strings by which the bodies A, B are supported, it is evident that, while the passenger moves them with respect to himself at speeds AD, BD, the one standing on the shore moves them with respect to himself and the shore at speeds AC, BC; since these speeds are in inverse proportion to their magnitudes, it is necessary that bodies A, B, with respect to the same spectator, rebound from contact at the same speeds CA, CB, as was shown in the preceding [propositions]. But the boat always proceeds at speed DC or CE, and it does so according to the order of the points C, E; therefore, it is necessary that A be moved with respect to the boat and the passenger at speed EA in the direction designated by the order of the points E, A. And B [is moved] with respect to the same boat at speed EB, again according to the order of the points E, B. When, moreover, E falls on A, or B, it is evident that body A, or B, is moved after collision at equal speed with this boat and in the same direction, whence the bodies in these cases are necessarily at rest with respect to the boat and the passenger. And so we have shown that bodies A and B, which in the boat were moved toward collision at speeds AD, BD, are moved after collision in the same boat at speeds EA, EB, according to the order of these points. But what obtains in the boat certainly results, as we have said, on motionless ground. Therefore, the proposition holds. For the purposes of calculation one may form from the construction of this problem the following rules.

If there be two bodies A and B, both of which are moved, to find the speed of body A after impulse, let the sum of the bodies be to twice body B as the speed they have with respect to each other is to another speed called C; the difference between this and the speed of body A before impulse, or in one case their sum (i.e. when A precedes in the motion), will give the speed at which this [body] will be moved after collision, rebounding if the excess is in favor of C and proceeding if on the contrary. And, if there is no difference, body A will be at rest after collision.

Moreover, when the speed of body A has been found, the speed of body B will also be known from the fact that the speed of the bodies with respect to one another should be the same before and after collision.

If body A be given at rest, and only B is moved toward it, it is evident that the speed of this A after collision will be equal to the speed C found in the manner we have just said. From this the following theorem may also be deduced.

# Proposition X

The speed that a greater body gives to a smaller one at rest has to that which the smaller body at a similar velocity impresses in the greater one at rest the same ratio that the magnitude of the greater [has] to the magnitude of the smaller.

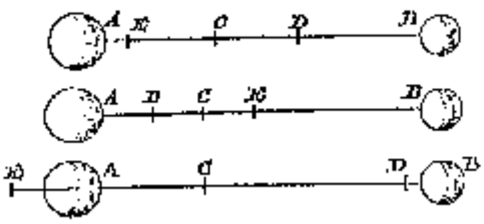
Let body A be greater than B and assume that, if A at rest is impelled by body B moved at velocity BA, A is accorded velocity AC, and, if B at rest is impelled by body A [moved] at an equal velocity AB, B is given velocity BD. I say that A is to B in magnitude as speed BD to AC.

For, because speed BD is to twice the speed AB as body A is to both B and A taken together, and also both B and A are to B as twice the speed AB to the speed AC, by equals speed BD will be to speed AC as body A to B; Q.E.D.

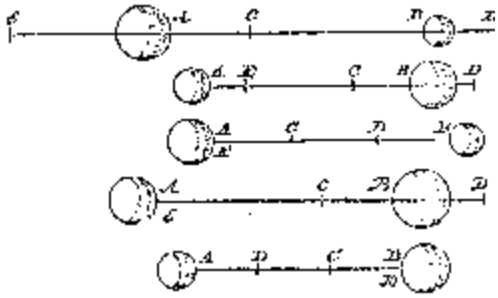
# Proposition XI

If two bodies collide with each other, that which results from multiplying the magnitudes of each by the square of their velocities, added together, is found to be equal before and after collision; if, that is, the ratios of both the magnitudes and the velocities are posited in numbers or lines.

Let the bodies be A and B, of which A is moved before collision at speed AD and B at speed BD. Be it also found, by the antecedent [proposition], that after collision the speed of body A is EA and the speed of body B, EB; namely, by dividing AB at C such that A is to B as BC to CA, and by



setting CE equal to CD. Because, therefore, the ratio of magnitude A to B is designated by the ratio of



line CB to CA, it must be shown that the solid composed of line CB multiplied by the square AD plus the solid composed of line CA multiplied by the square BD is equal to the sum of the solid composed of the same CB multiplied by the square EA and the solid composed of line CA multiplied by the square EB. But, if there are four magnitudes, of which the first exceeds the second by as much as the third [exceeds] the fourth, or of which the first is less than the second by as much as the third [is less than] the fourth, then it is certain that the first plus the fourth is equal to the second plus the third. Thus, the proposition will hold, if we show that the solid from square AD times line CB exceeds, or is exceeded by, the solid from square EA times the same CB by as much as that which results from square EB times line CA at the same time exceeds, or is exceeded by, the solid from square BD times line CA. And this is shown as follows.

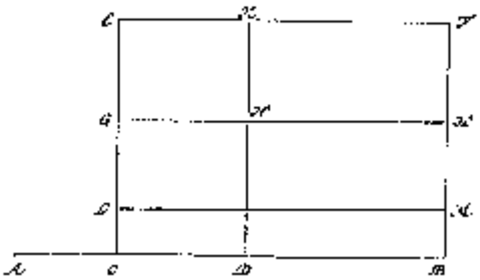
In every case, either point C falls between A and D, or D between A and C. Whenever C is located between A and D, AD will be equal to the two AC, CD taken together, and AE to their difference. Now CE is equal to CD, whence AD will always be greater than AE. In the same cases, BE will be equal to the two BC, CE taken together, and BD to their difference; and again BE will always be greater than BD. But whenever D falls between A and C, AE will be equal to the two AC, CE taken together, and AD to their difference; and consequently AE will be greater than AD. But also in these cases BD will be greater than BE, since the former will be equal to the two BC, CD taken together and the latter to their difference. Thus it is clear that, whenever AD is greater than AE, BE is also greater than BD; and, whenever AE is greater than AD, BD will also be greater than BE.

Next, since DE is divided into equals at C, whatever the position of point A, the difference of the square AD, AE will always be equal to four times the rectangle ACD, or ACE, by the eighth [proposition] of the second [book] of [Euclid's] *Elements* (taking for the line cut in any way: in the first and fifth case AC, which is divided at E; in the second and eighth AC, which is divided at D; in the third and fourth case EC, which A cuts; in the sixth and seventh, where there is no square AE, it is clear that the said difference is square AD, which similarly is equal to four times the rectangle ACD, or ACE). For the same reason, in the case of the bisection of line DE at C, whatever the position of point B, the difference of the squares BE, BD will always be equal to four times the rectangle BCD, or BCE. But, due to the common altitude, four times the rectangle BCD is to four times the rectangle ACD, which was equal to the difference of the squares AD, AE as BC to AC. Therefore, the difference of the squares BE, BD is to the difference of the squares AD, AE as BC to AC. Wherefore, the product of the difference of the squares AD, AE times the line BC, which [product] is itself the difference of the solids from square AD times BC and from square AE times BC, is equal to the product of the difference of the squares BE, BD times the line AC, i.e. the difference of the solids from the square BE times AC and from the square BD times AC.

But, when square AD exceeds, or is less than, square AE, then square BE also always at the same time exceeds, or is exceeded by square BD. Therefore, it is clear that the solid from the square AD times BC always exceeds, or is exceeded by, the product of square AE times BC by as much as the product of square BE times AC at the same time exceeds, or is exceeded by, the product of square BD times AC; Q.E.D.

# Lemma I

Let the straight line AB be cut at C and D such that the segment AC is less than CD, and CD less than BD. I say that the rectangle [formed] from AD, CB is less than twice both rectangles ACD, CDB taken together.

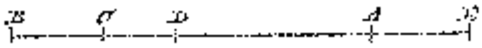


Describe on segment CD the square CGND, and produce CG to E such that GE is equal to CA, and complete rectangle ECBF, and produce DN to K, GN to H. Since, therefore, CG is equal to CD, and GE equal to AC, the whole CE will be equal to AD. Thus, rectangle CF is that contained by AD, CB. Rectangle EN is equal to rectangle ACD, and rectangle NB is equal to rectangle CDB. One must show, therefore, that rectangle CF is less than twice the rectangles EN and NB taken together. Take GL equal to GE, and draw LM parallel to AB. Because GL is less than GC (for GC, or AC, is less than CD), LM falls between GH and CB.

Then, because CD is less than DB, rectangle LD will be less than rectangle DM. But LN is equal to rectangle NE, and rectangle NM is equal to rectangle NF, and hence the two rectangles LN and NF together are equal to the two NE and NM, and thus, if equals are added to unequals, that is to rectangles LN and NF [is added] rectangle LD and to rectangles NE, NM rectangle MD, what is composed of the former, i.e. square CN plus rectangle NF, is less than what is composed of the latter, i.e. rectangle NB plus rectangle NE. Whence what is composed of all taken together, i.e. rectangle CF, is clearly less than twice rectangles NB and NE; Q.E.D.

# Lemma II

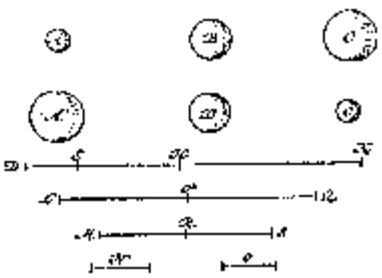
Let AB, AC, AD be three proportional straight lines, of which AB is the larger, and let the same length AE be added to each of them. I say the rectangle formed by BE, DE is greater than the square CE.



For, because AB, AC, AD are proportional, the excess BC will also be to the excess CD as BA to AC, or as CA to AD. But the ratio CA to AD is greater than CE to ED, and thus BC to CD is also greater than CE to ED, and, by permutation, the ratio BC to CE is greater than CD to DE; by composition, therefore, the ratio BE to EC is greater than CE to ED. Wherefore the rectangle formed by BE, ED is greater than the square of CE; which was proposed.

# Proposition XII

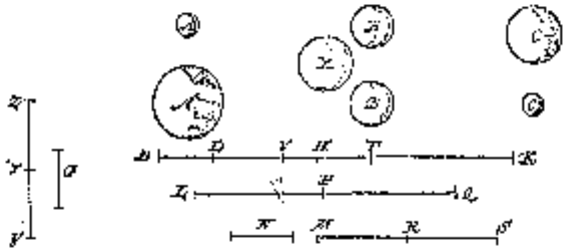
If a body moves toward a greater or smaller body at rest, it will give it a greater speed through an interposed body of intermediate magnitude, also at rest, than if it strikes it through no intermediate. And it will confer the maximum speed when the interposed body is the mean proportional between the extremes.



Let body A be moved toward C, which is at rest, and let A be greater or less than this C, and let body B be placed in between them, at rest and of intermediate magnitude, such that A first impels B and B then impels C. I say that greater motion is thus acquired by body C than if A simply collided with it.

Let the straight lines DE, EH, HK have among them the same ratio that the bodies A, B, C have among themselves, and let LP be the speed of body A, and let the double of [this speed] be LQ. If, therefore, both DE, EH taken together are to DE as LQ to MR, MR will be the speed acquired by the body B at rest when it is impelled by A. Let MS be twice this MR. If, furthermore, both EH, HK taken together are to EH as MS to N, then N will be the speed of body C after it is pushed by B at speed MR. And, if DE, HK taken together are to DE as LQ to 0, 0 will be the speed sought of body C if it is impelled by body A at speed LP. Thus one must demonstrate that speed N is greater than O.

The ratio LQ to N is composed of the ratios LQ to MR and MR to N. But the ratio LQ to MR is the same as the ratio HD to DE. And the ratio MR to N is the same as the ratio KE to twice EH. For as KE is to EH, so SM is to N; whence KE is to twice EH as SM to 2N, that is, as RM to N. Hence, the ratio LQ to N is composed of the ratios HD to DE and KE to twice EH, and consequently it will be that of rectangle HD, KE to twice rectangle DEH. But the ratio LQ to O is that which both DE, HK taken together have to DE by construction, that is, given the common altitude EH, [the ratio is that] which the two rectangles DEH, EHK taken together have to rectangle DEH, or which the former taken twice [have] to twice rectangle DEH. But rectangle HD, KE is less than twice rectangles DEH, EHK. Hence the ratio of rectangle HD, KE to twice rectangle DEH will be less than that of rectangles DEH, EHK taken twice to the same double rectangle DEH. But LQ to N has been said to have the same ratio that rectangle HD, KE has to twice rectangle DEH. And LQ to O has been said to have the same ratio that twice rectangles DEH, EHK have to twice rectangle DEH. Therefore, the ratio LQ to N will be less than LQ to 0, and consequently N will be greater than O.



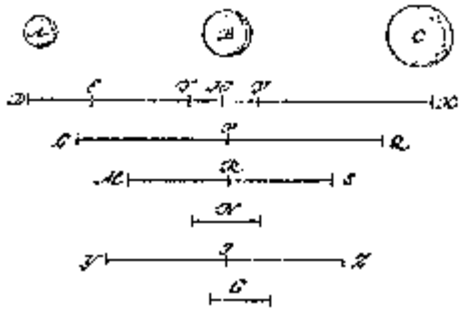
Now let B be proportionately intermediate between A and C; I say that by this ratio the greatest of all speeds will be given to body C.

For, if it can be done, interpose first in place of B a greater body X such that A impels X, and X impels C, and let it be said that a greater speed is thus acquired by C than if B were interposed. And let DE be to ET as A is to X. Hence, ET is greater than EH, on the assumption, that is, that, as before, DE, EH, HK are proportionals in the same ratio as the bodies A, B, C. Also, let VE be a third proportional to the two TE, HE, and consequently, as in the preceding [cases] the speed N acquired by body C through the interposed [body] B is found. By a similar argument is found the speed acquired by the same C through the interposed [body] X. That is, if LQ is to IY as both A and X taken together are to X, i.e. as both DE, ET taken together are to DE, then IY will be the speed impressed in body X by the impelling A. Whence if, further, twice IY, which ZY, is to G as both X and C taken together are to X, i.e. as both ET, HK taken together are to ET, then G will be the speed sought in body C. Wherefore one must show that N is greater than G.

The ratio LQ to N, as before, is shown to be composed of ratios HD to DE and KE to twice HE. But KE is to twice HE as HD to twice ED, because KH, HE, ED are proportional. Therefore the ratio LQ to N is composed now of ratios HD to DE and HD to twice DE, and for that reason it will be the same as that of square HD to twice square DE. But the ratio LQ to G is composed of ratios LQ to IY and IY to G, of which the ratio LQ to IY is the same as that of TD to DE by construction, and the ratio IY to G the same as that of both KH, TE taken together to twice TE; for, by construction, both KH, TE taken together are to TE as ZY to G, and hence, taking double consequents, as the two KH, TE are to twice TE so ZY is to twice G, or IY to G, as has been said. Thus the ratio LQ to G is composed of the ratios TD to DE and the two KH, TE taken together to twice TE. And because DE, EH, HK are proportional, the rectangle DE, HK will be equal to the square EH. But the rectangle EV, ET is also equal to the same square EH, since EV, EH, ET are proportional. Therefore rectangle DE, HK is equal to rectangle EV, ET. Whence, VE is to ED as HK is to ET; by composition, VD is to DE as both KH, TE taken together are to TE, and, taking double consequents, VD is to twice DE as the two HK, TE are to twice TE. Thus the ratio LQ to G is composed of the ratios TD to DE and VD to twice DE, and consequently it is the same as that of rectangle TDV to twice square DE. But the ratio LQ to N has been shown to be the same as that of square HD to twice square DE. Thus, since rectangle TDV is greater than square HD by Lemma II (for TE, HE, VE are proportionals to which the length ED has been added), it follows that the ratio LQ to G is greater than LQ to N, hence N is greater than G, which was to be shown.



Let it then be said that by the interposition of a body X less



than B a greater speed is acquired by body C. Further, let A be to X as DE to ET. Therefore, because X is now assumed to be less than B, ET will also be less than EH, for A is to B as DE to EH. But for the rest one repeats the construction and demonstration that has already been set forth, by which speed N is again shown to be greater than G. Thus it holds that the maximum speed is acquired by the body C at rest through the interposition of a body B that is the mean proportional between A and C.

# Proposition XIII

According as more bodies are interposed between two unequal bodies of which one is at rest and one is moved, a greater motion will be yielded to the body at rest. And the greatest [motion] will be conferred by any one number of interposed [bodies] if the interposed [bodies] are such that they constitute with the extremes a continuous series of proportionals.

Let bodies A, B, C be proportionals, of which A is moved [and] the remaining two are at rest. The maximum motion to be acquired by body C through the interposition of one body is that which is caused by the interposed [body] B. But that a still greater [motion] can be caused with the aid of two interposed bodies will be proved from the following. For if, between A and B, a mean proportional D is interposed, a greater motion will now be acquired by body B than if it were simply struck by body A. But the greater the speed in B, so also a greater speed is induced in C. Therefore C will be moved more through the interposition of bodies D, B than through B alone. But if, in place of B, another body were then set up that was the mean proportional between D and C, it is evident that an even greater motion would be transferred to C than through the interposed bodies D, B. Next, that the maximum motion is yielded to the extreme body by the interposition of two bodies when A, D, B, C are continuously proportional is shown thus. To begin with, it holds that the speed of body C cannot grow to any size [*in quantumvis magnam*] through the interposition of two bodies. For the speed of body D will always be less than twice the speed of A. Furthermore, the speed of B will always be less than twice the speed of D, and the speed acquired by body C always will be less than twice the speed of B. Hence the speed of C will at least have to be less than eight times the speed of A. From this it is thus understood that some definite speed exists greater than which cannot be acquired by body C through the interposition of two bodies. Let E be that speed, which we suppose moreover to have been acquired by body C with bodies D, B interposed between it and A. I say that A, D, B, C are continuously proportional. For, first, if the three A, D, B are not proportional, substitute for body D another body which is the mean proportional between A and B, such that a greater motion goes over to B than through the interposed D. Consequently, C will acquire a greater velocity than through the interposed D, B, i. e. greater than velocity E, which is absurd because it was posited that E is the maximum velocity that body C could acquire by the interposition of two bodies. Similarly, if D~ B, C are not proportional, one can set up in the place of B another mean proportional between D, C, such that again a greater velocity would be acquired by body E, which is absurd for the same reason. Thus, since both A, D, B and D, B, C are proportional, all the bodies A, D, B, C will be in continuous proportion, which was to be shown. From this it can now be shown by the same argument that, if three bodies were interposed between A, C, a still greater motion could be given to body C than when only two are interposed, and so on. And by a similar argument, the maximum motion is shown to be acquired when the proportion of all bodies is continuous. Thus the proposition holds.

If one hundred bodies in double proportion are given in order, and the motion begins from the greatest, one finds by carrying out the calculation according to the precept of the rule set forth in proposition IX, but abbreviated in the compendium, that the speed of the smallest body is to the speed at which the greatest is moved approximately as 14,760,000,000 to 1. And if the motion begins from the smallest, the quantity of motion grows into the universe according to the ratio approximately 1 to 4,677,000,000,000.