HEINRICH HERTZ

THE

PRINCIPLES OF MECHANICS

PRESENTED IN A NEW FORM

Preface by H. VON HELMHOLTZ; *authorized English translation by*

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Dover Publications, Inc., New York

to make the further hypothesis that there are a large number of imperceptible masses with invisible motions, in order to explain the existence of forces between bodies which are not in direct contact with each other. Unfortunately he has not given examples illustrating the manner in which he supposed such hypothetical mechanism to act; to explain even the simplest cases of physical forces on these lines will clearly require much scientific insight and imaginative power. In this direction Hertz seems to have relied chiefly on the introduction of cyclical systems with invisible motions.

English physicists—*e.g.* Lord Kelvin, in his theory of vortex-atoms, and Maxwell, in his hypothesis of systems of cells with rotating contents, on which he bases his attempt at a mechanical explanation of electromagnetic processes—have evidently derived a fuller satisfaction from such explanations than from the simple representation of physical facts and laws in the most general form, as given in systems of differential equations. For my own part, I must admit that I have adhered to the latter mode of representation and have felt safer in so doing; yet I have no essential objections to raise against a method which has been adopted by three physicists of such eminence.

It is true that great difficulties have yet to be overcome before we can succeed in explaining the varied phenomena of physics in accordance with the system developed by Hertz. But in every respect his presentation of the *Principles of Mechanics* is a book which must be of the greatest interest to every reader who can appreciate a logical system of dynamics developed with the greatest ingenuity and in the most perfect mathematical form. In the future this book may prove of great heuristic value as a guide to the discovery of new and general characteristics of natural forces.

AUTHOR'S PREFACE

ALL physicists agree that the problem of physics consists in tracing the phenomena of nature back to the simple laws of mechanics. But there is not the same agreement as to what these simple laws are. To most physicists they are simply Newton's laws of motion. But in reality these latter laws only obtain their inner significance and their physical meaning through the tacit assumption that the forces of which they speak are of a simple nature and possess simple properties. But we have here no certainty as to what is simple and permissible, and what is not: it is just here that we no longer find any general agreement. Hence there arise actual differences of opinion as to whether this or that assumption is in accordance with the usual system of mechanics, or not. It is in the treatment of new problems that we recognise the existence of such open questions as a real bar to progress. So, for example, it is premature to attempt to base the equations of motion of the ether upon the laws of mechanics until we have obtained a perfect agreement as to what is understood by this name.

The problem which I have endeavoured to solve in the present investigation is the following:—To fill up the existing gaps and to give a complete and definite presentation of the laws of mechanics which shall be consistent with the state of our present knowledge, being neither too restricted nor too extensive in relation to the scope of this knowledge. The presentation must not be too restricted: there must be no natural motion which it does not embrace. On the other hand it must not be too extensive: it must admit of no motion whose occurrence in nature is excluded by the state of our present knowledge. Whether the presentation here given as the solution of this problem is the only possible one, or whether there are other and perhaps better possible ones, remains open. But that the presentation given is in every respect a possible one, I prove by developing its consequences, and showing that when fully unfolded it is capable of embracing the whole content of ordinary mechanics, so far as the latter relates only to the actual forces and connections of nature, and is not regarded as a field for mathematical exercises.

In the process of this development a theoretical discussion has grown into a treatise which contains a complete survey of all the more important general propositions in dynamics, and which may serve as a systematic text-book of this science. For several reasons it is not well suited for use as a first introduction; but for these very reasons it is the better suited to guide those who have already a fair mastery of mechanics as usually taught. It may lead them to a vantage-ground from which they can more clearly perceive the physical meaning of mechanical principles, how they are related to each other, and how far they hold good; from which the ideas of force and the other fundamental ideas of mechanics appear stripped of the last remnant of obscurity.

In his papers on the principle of least action and on cyclical systems,[1] von Helmholtz has already treated in an indirect manner the problem which is investigated in this book, and has given one possible solution of it. In the first set of papers he propounds and maintains the thesis that a system of mechanics which regards as of universal validity, not only Newton's laws, but also the special assumptions involved (in addition to these laws) in Hamilton's Principile,

[1] H. von Helmholtz, "Über die physikalische Bedeutung des Prinzips der kleinsten Wirkung," *Journal für die reine und angewandte Mathematik*, **100**, pp. 137-166, 213-222, 1887; "Prinzipien der Statik monocyklischer Systeme," *ibid.* **97**, pp. 111-140, 317-336, 1884.

would yet be able to embrace all the processes of nature. In the second set of papers the meaning and importance of concealed motions is for the first time treated in a general way. Both in its broad features and in its details my own investigation owes much to the above-mentioned papers: the chapter on cyclical systems is taken almost directly from them. Apart from matters of form, my own solution differs from that of von Helmholtz chiefly in two respects. Firstly, I endeavour from the start to keep the elements of mechanics free from that which von Helmholtz only removes by subsequent restriction from the mechanics previously developed. Secondly, in a certain sense I eliminate less from mechanics, inasmuch as I do not rely upon Hamilton's Principle or any other integral principle. The reasons for this and the consequences which arise from it are made clear in the book itself.

In his important paper on the physical applications of dynamics, J. J. Thomson[1] pursues a train of thought similar to that contained in von Helmholtz's papers. Here again the author develops the consequences of a system of dynamics based upon Newton's laws of motion and also upon other special assumptions which are not explicitly stated. I might have derived assistance from this paper as well; but as a matter of fact my own investigation had made considerable progress by the time I became familiar with it. I may say the same of the mathematical papers of Beltrami[2] and Lipschitz,[3] although these are of much older date. Still I found these very suggestive, as also the more recent presentation of their investigations which Darboux[4] has given with

[1] J. J. Thomson, "On some Applications of Dynamical Principles to Physical Phenomena," *Philosophical Transactions*, **176**, II., pp. 307-342, 1885.

[2] Beltrami, "Sulla teoria generale dei parametri differenziali," *Memorie della Reale Accademia di Bològna*, 25 Febbrajo 1869.

[3] R. Lipschitz, "Untersuchungen eines Problems der Variationsrechnung, in welchem das Problem der Mechanik enthalten ist," *Journal für die reine und angewandte Mathematik*, **74**, pp. 116-149, 1872. "Bemerkungen zu dem Princip des kleinsten Zwanges," *ibid.* **82**, pp. 316-342, 1877.

[4] G. Darboux, *Leçons sur la théorie générale des surfaces*, livre v., chapitres vi. vii. viii., Paris, 1889.

additions of his own. I may have missed many mathematical papers which I could and should have consulted. In a general way I owe very much to Mach's splendid book on the *Development of Mechanics.*[1] I have naturally consulted the better-known text-books of general mechanics, and especially Thomson and Tait's comprehensive treatise.[2] The notes of a course of lectures on analytical dynamics by Borchardt, which I took down in the winter of 1878-79, have proved useful. These are the sources upon which I have drawn; in the text I shall only give such references as are requisite. As to the details I have nothing to bring forward which is new or which could not have been gleaned from many books. What I hope is new, and to this alone I attach value, is the arrangement and collocation of the whole—the logical or philosophical aspect of the matter. According as it marks an advance in this direction or not, my work will attain or fail of its object.

[1] E. Mach, *Die Mechanik in ihrer Entwickelung historisch-kritisch dargestellt*, Leipzig, 1883 (of this there is an English translation by T. J. M'Cormack, *The Science of Mechanics*, Chicago, 1893).

[2] Thomson and Tait, *Natural Philosophy*.

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CHAPTER VI

INTRODUCTION

THE most direct, and in a sense the most important, problem which our conscious knowledge of nature should enable us to solve is the anticipation of future events, so that we may arrange our present affairs in accordance with such anticipation. As a basis for the solution of this problem we always make use of our knowledge of events which have already occurred, obtained by chance observation or by prearranged experiment. In endeavouring thus to draw inferences as to the future from the past, we always adopt the following process. We form for ourselves images or symbols of external objects; and the form which we give them is such that the necessary consequents of the images in thought are always the images of the necessary consequents in nature of the things pictured. In order that this requirement may be satisfied, there must be a certain conformity between nature and our thought. Experience teaches us that the requirement can be satisfied, and hence that such a conformity does in fact exist. When from our accumulated previous experience we have once succeeded in deducing images of the desired nature, we can then in a short time develop by means of them, as by means of models, the consequences which in the external world only arise in a comparatively long time, or as the result of our own interposition. We are thus enabled to be in advance of the facts, and to decide as to present affairs in accordance with the insight so obtained. The images which we here speak of are our conceptions of things. With the things themselves they are in conformity in *one* important respect, namely, in satisfying the above-mentioned requirement. For our purpose it is not necessary that they should be in conformity with the things in any other respect whatever. As a matter of fact, we do not know, nor have we any means of knowing, whether our conceptions of things are in conformity with them in any other than this *one* fundamental respect.

The images which we may form of things are not determined without ambiguity by the requirement that the consequents of the images must be the images of the consequents. Various images of the same objects are possible, and these images may differ in various respects. We should at once denote as inadmissible all images which implicitly contradict the laws of our thought. Hence we postulate in the first place that all our images shall be logically permissible—or, briefly, that they shall be permissible. We shall denote as incorrect any permissible images, if their essential relations contradict the relations of external things, *i.e.* if they do not satisfy our first fundamental requirement. Hence we postulate in the second place that our images shall be correct. But two permissible and correct images of the same external objects may yet differ in respect of appropriateness. Of two images of the same object that is the more appropriate which pictures more of the essential relations of the object,—the one which we may call the more distinct. Of two images of equal distinctness the more appropriate is the one which contains, in addition to the essential characteristics, the smaller number of superfluous or empty relations,—the simpler of the two. Empty relations cannot be altogether avoided: they enter into the images because they are simply images,—images produced by our mind and necessarily affected by the characteristics of its mode of portrayal.

The postulates already mentioned are those which we assign to the images themselves: to a scientific representation of the images we assign different postulates. We require of this that it should lead us to a clear conception of what properties are to be ascribed to the images for the sake of permissibility, what for correctness, and what for appropriatenesss. Only thus can we attain the possibility of modifying and improving our images. What is ascribed to the

images for the sake of appropriateness is contained in the notations, definitions, abbreviations, and, in short, all that we can arbitrarily add or take away. What enters into the images for the sake of correctness is contained in the results of experience, from which the images are built up. What enters into the images, in order that they may be permissible, is given by the nature of our mind. To the question whether an image is permissible or not, we can without ambiguity answer yes or no; and our decision will hold good for all time. And equally without ambiguity we can decide whether an image is correct or not; but only according to the state of our present experience, and permitting an appeal to later and riper experience. But we cannot decide without ambiguity whether an image is appropriate or not; as to this differences of opinion may arise. One image may be more suitable for one purpose, another for another; only by gradually testing many images can we finally succeed in obtaining the most appropriate.

Those are, in my opinion, the standpoints from which we must estimate the value of physical theories and the value of the representations of physical theories. They are the standpoints from which we shall here consider the representations which have been given of the Principles of Mechanics. We must first explain clearly what we denote by this name.

Strictly speaking, what was originally termed in mechanics a principle was such a statement as could not be traced back to other propositions in mechanics, but was regarded as a direct result obtained from other sources of knowledge. In the course of historical development it inevitably came to pass that propositions, which at one time and under special circumstances were rightly denoted as principles, wrongly retained these names. Since Lagrange's time it has frequently been remarked that the principles of the centre of gravity and of areas are in reality only propositions of a general nature. But we can with equal justice say that other so-called principles cannot bear this name, but must descend to the rank of propositions or corollaries, when the representation of mechanics becomes based upon one or more of the others. Thus the idea of a mechanical principle has not been kept sharply defined. We shall therefore retain for such propositions, when mentioning them separately, their customary names. But these separate concrete propositions are not what we shall have in mind when we speak simply and generally of the principles of mechanics: by this will be meant any selection from amongst such and similar propositions, which satisfies the requirement that the whole of mechanics can be developed from it by purely deductive reasoning without any further appeal to experience. In this sense the fundamental ideas of mechanics, together with the principles connecting them, represent the simplest image which physics can produce of things in the sensible world and the processes which occur in it. By varying the choice of the propositions which we take as fundamental, we can give various representations of the principles of mechanics. Hence we can thus obtain various images of things; and these images we can test and compare with each other in respect of permissibility, correctness, and appropriateness.

I

The customary representation of mechanics gives us a first image. By this we mean the representation, varying in detail but identical in essence, contained in almost all text-books which deal with the whole of mechanics, and in almost all courses of lectures which cover the whole of this science. This is the path by which the great army of students travel and are inducted into the mysteries of mechanics. It closely follows the course of historical development and the sequence of discoveries. Its principal stages are distinguished by the names of Archimedes, Galileo, Newton, Lagrange. The conceptions upon which this representation is based are the ideas of space, time, force, and mass. In it force is introduced as the cause of motion, existing before motion and independently of it. Space and force first appear by themselves, and their relations are treated of in statics. Kinematics, or the science of pure motion, confines itself to connecting the two ideas of space and time. Galileo's conception of inertia furnishes a connection between space, time, and mass alone. Not until Newton's Laws of Motion do the four fundamental ideas

become connected with each other. These laws contain the seed of future developments; but they do not furnish any general expression for the influence of rigid spacial connections. Here d'Alembert's principle extends the general results of statics to the case of motion, and closes the series of independent fundamental statements which cannot be deduced from each other. From here on everything is deductive inference. In fact the above-mentioned ideas and laws are not only necessary but sufficient for the development of the whole of mechanics from them as a necessary consequence of thought; and all other so-called principles can be regarded as propositions and corollaries deduced by special assumptions. Hence the above ideas and laws give us, in the sense in which we have used the words, a first system of principles of mechanics, and at the same time the first general image of the natural motions of material bodies.

Now, at first sight, any doubt as to the logical permissibility of this image may seem very far-fetched. It seems almost inconceivable that we should find logical imperfections in a system which has been thoroughly and repeatedly considered by many of the ablest intellects. But before we abandon the investigation on this account, we should do well to inquire whether the system has always given satisfaction to these able intellects. It is really wonderful how easy it is to attach to the fundamental laws considerations which are quite in accordance with the usual modes of expression in mechanics, and which yet are an undoubted hindrance to clear thinking. Let us endeavour to give an example of this. We swing in a circle a stone tied to a string, and in so doing we are conscious of exerting a force upon the stone. This force constantly deflects the stone from its straight path. If we vary the force, the mass of the stone, and the length of the string, we find that the actual motion of the stone is always in accordance with Newton's second law. But now the third law requires an opposing force to the force exerted by the hand upon the stone. With regard to this opposing force the usual explanation is that the stone reacts upon the hand in consequence of centrifugal force, and that this centrifugal force is in fact exactly equal and opposite to that which we exert. Now is this mode of expression permissible? Is what we call centrifugal force anything else than the inertia of the stone? Can we, without destroying the clearness of our conceptions, take the effect of inertia twice into account,—firstly as mass, secondly as force? In our laws of motion, force was a cause of motion, and was present *before* the motion. Can we, without confusing our ideas, suddenly begin to speak of forces which arise through motion, which are a consequence of motion? Can we behave as if we had already asserted anything about forces of this new kind in our laws, as if by calling them forces we could invest them with the properties of forces? These questions must clearly be answered in the negative. The only possible explanation is that, properly speaking, centrifugal force is not a force at all. Its name, like the name *vis viva*, is accepted as a historic tradition; it is convenient to retain it, although we should rather apologise for its retention than endeavour to justify it. But, what now becomes of the demands of the third law, which requires a force exerted by the inert stone upon the hand, and which can only be satisfied by an actual force, not a mere name?

I do not regard these as artificial difficulties wantonly raised: they are objections which press for an answer. Is not their origin to be traced back to the fundamental laws? The force spoken of in the definition and in the first two laws acts upon a body in one definite direction. The sense of the third law is that forces always connect two bodies, and are directed from the first to the second as well as from the second to the first. It seems to me that the conception of force assumed and created in us by the third law on the one hand, and the first two laws on the other hand, are slightly different. This slight difference may be enough to produce the logical obscurity of which the consequences are manifest in the above example. It is not necessary to discuss further examples. We can appeal to general observations as evidence in support of the above-mentioned doubt.

As such, in the first place, I would mention the experience that it is exceedingly difficult to expound to thoughtful hearers the very introduction to mechanics without being occasionally embarrassed, without feeling tempted now and again to apologise, without wishing to get as quickly as possible over the rudiments, and on to examples which speak for themselves. I fancy that Newton himself must have felt this embarrassment when he gave the rather forced definition of mass as being the product of volume and density. I fancy that Thomson and Tait must also have felt it when they remarked that this is really more a definition of density than of mass, and nevertheless contented themselves with it as the only definition of mass. Lagrange, too, must have felt this embarrassment and the wish to get on at all costs; for he briefly introduces his Mechanics with the explanation that a force is a cause which imparts "or tends to impart" motion to a body; and he must certainly have felt the logical difficulty of such a definition. I find further evidence in the demonstrations of the elementary propositions of statics, such as the law of the parallelogram of forces, of virtual velocities, etc. Of such propositions we have numerous proofs given by eminent mathematicians. These claim to be rigid proofs; but, according to the opinion of other distinguished mathematicians, they in no way satisfy this claim. In a logically complete science, such as pure mathematics, such a difference of opinion is utterly inconceivable.

Weighty evidence seems to be furnished by the statements which one hears with wearisome frequency, that the nature of force is still a mystery, that one of the chief problems of physics is the investigation of the nature of force, and so on. In the same way electricians are continually attacked as to the nature of electricity. Now, why is it that people never in this way ask what is the nature of gold, or what is the nature of velocity? Is the nature of gold better known to us than that of electricity, or the nature of velocity better than that of force? Can we by our conceptions, by our words, completely represent the nature of any thing? Certainly not. I fancy the difference must lie in this. With the terms "velocity" and "gold" we connect a large number of relations to other terms; and between all these relations we find no contradictions which offend us. We are therefore satisfied and ask no further questions. But we have accumulated around the terms "force" and "electricity" more relations than can be completely reconciled amongst themselves. We have an obscure feeling of this and want to have things cleared up. Our confused wish finds expression in the confused question as to the nature of force and electricity. But the answer which we want is not really an answer to this question. It is not by finding out more and fresh relations and connections that it can be answered; but by removing the contradictions existing between those already known, and thus perhaps by reducing their number. When these painful contradictions are removed, the question as to the nature of force will not have been answered; but our minds, no longer vexed, will cease to ask illegitimate questions.

I have thrown such strong doubts upon the permissibility of this image that it might appear to be my intention to contest, and finally to deny, its permissibility. But my intention and conviction do not go so far as this. Even if the logical uncertainties, which have made us solicitous as to our fundamental ideas, do actually exist, they certainly have not prevented a single one of the numerous triumphs which mechanics has won in its applications. Hence, they cannot consist of contradictions between the essential characteristics of our image, nor, therefore, of contradictions between those relations of mechanics which correspond to the relations of things. They must rather lie in the unessential characteristics which we have ourselves arbitrarily worked into the essential content given by nature. If so, these dilemmas can be avoided. Perhaps our objections do not relate to the content of the image devised, but only to the form in which the content is represented. It is not going too far to say that this representation has never attained scientific completeness; it still fails to distinguish thoroughly and sharply between the elements in the image which arise from the necessities of thought, from experience, and from arbitrary choice. This is also the opinion of distinguished physicists who have thought over and discussed [1] these questions, although it cannot be said that all of them are in agreement.[2] This opinion also finds confirmation in the increasing care with which the logical analysis of the elements is carried out in the more recent text-books of mechanics.[3] We are con-

[1] See E. Mach, *The Science of Mechanics*, p. 244. See also in *Nature* (**48**, pp. 62, 101, 117, 126 and 166, 1893; and *Proc. Phys. Soc.* **12**, p. 289, 1893) a discussion on the foundations of dynamics introduced by Prof. Oliver Lodge and carried

vinced, as are the authors of these text-books and the physicists referred to, that the existing defects are only defects in form; and that all indistinctness and uncertainty can be avoided by suitable arrangement of definitions and notations, and by due care in the mode of expression. In this sense we admit, as everyone does, the permissibility of the content of mechanics. But the dignity and importance of the subject demand, not simply that we should readily take for granted its logical clearness, but that we should endeavour to show it by a representation so perfect that there should no longer be any possibility of doubting it.

Upon the correctness of the image under consideration we can pronounce judgment more easily and with greater certainty of general assent. No one will deny that within the whole range of our experience up to the present the correctness is perfect; that all those characteristics of our image, which claim to represent observable relations of things, do really and correctly correspond to them. Our assurance, of course, is restricted to the range of previous experience: as far as future experience is concerned, there will yet be occasion to return to the question of correctness. To many this will seem to be excessive and absurd caution: to many physicists it appears simply inconceivable that any further experience whatever should find anything to alter in the firm foundations of mechanics. Nevertheless, that which is derived from experience can again be annulled by experience. This over-favourable opinion of the fundamental laws must obviously arise from the fact that the elements of experience are to a certain extent hidden in them and blended with the unalterable elements which are necessary consequences of our thought. Thus the logical indefiniteness of the representation, which we have just censured, has one advantage. It gives the foundations an appearance of immutability; and perhaps it was wise to introduce it in the beginnings of the science and to allow it to remain for a while. The correctness of the image in all cases was carefully provided for by making the reservation that, if need be, facts derived from experience should determine definitions or *vice versa*. In a perfect science such groping, such an appearance of certainty, is inadmissible.

Mature knowledge regards logical clearness as of prime importance: only logically clear images does it test as to correctness; only correct images does it compare as to appropriateness. By pressure of circumstances the process is often reversed. Images are found to be suitable for a certain purpose; are next tested as to their correctness; and only in the last place purged of implied contradictions.

If there is any truth in what we have just stated, it seems only natural that the system of mechanics under consideration should prove most appropriate in its applications to those simple phenomena for which it was first devised, *i.e.* especially to the action of gravity and the problems of practical mechanics. But we should not be content with this. We should remember that we are not here representing the needs of daily life or the standpoint of past times; we are considering the whole range of present physical knowledge, and are, moreover, speaking of appropriateness in the special sense defined in the beginning of this introduction. Hence we are at once bound to ask,—Is this image perfectly distinct? Does it contain all the characteristics which our present knowledge enables us to distinguish in natural motions? Our answer is a decided—No. All the motions of which the fundamental laws admit, and which are treated of in mechanics as mathematical exercises, do not occur in nature. Of natural motions, forces, and fixed connections, we can predicate more than the accepted fundamental laws do. Since the middle of this century we have been firmly convinced that no forces actually exist in nature which would involve a violation of the principle of the conservation of energy. The conviction is much older that only such forces exist as can be represented as a sum of mutual actions between infinitely small elements of matter. Again, these elementary forces are not free. We can assert as a property which they are generally admitted to possess, that they are independent of absolute time and place. Other properties are disputed. Whether the elementary forces can only consist of attractions and repulsions along the line connecting the acting masses; whether their magnitude is determined only by the distance or whether it is also affected by the absolute or relative velocity;

whether the latter alone comes into consideration, or the acceleration or still higher differential coefficients as well—all these properties have been sometimes presumed, at other times questioned. Although there is such difference of opinion as to the precise properties which are to be attributed to the elementary forces, there is a general agreement that more of such general properties can be assigned, and can from existing observations be deduced, than are contained in the fundamental laws. We are convinced that the elementary forces must, so to speak, be of a simple nature. And what here holds for the forces, can be equally asserted of the fixed connections of bodies which are represented mathematically by equations of condition between the coordinates and whose effect is determined by d'Alembert's principle. It is mathematically possible to write down any finite or differential equation between coordinates and to require that it shall be satisfied; but it is not always possible to specify a natural, physical connection corresponding to such an equation: we often feel, indeed sometimes are convinced, that such a connection is by the nature of things excluded. And yet, how are we to restrict the permissible equations of condition? Where is the limiting line between them and the conceivable ones? To consider only finite equations of condition, as has often been done, is to go too far; for differential equations which are not integrable can actually occur as equations of condition in natural problems.

In short, then, so far as the forces, as well as the fixed relations, are concerned, our system of principles embraces all the natural motions; but it also includes very many motions which are not natural. A system which excludes the latter, or even a part of them, would picture more of the actual relations of things to each other, and would therefore in this sense be more appropriate. We are next bound to inquire as to the appropriateness of our image in a second direction. Is our image simple? Is it sparing in unessential characteristics—ones added by ourselves, permissibly and yet arbitrarily, to the essential and natural ones? In answering this question our thoughts again turn to the idea of force. It cannot be denied that in very many cases the forces which are used in mechanics for treating physical problems are simply sleeping partners, which keep out of the business altogether when actual facts have to be represented. In the simple relations with which mechanics originally dealt, this is not the case. The weight of a stone and the force exerted by the arm seem to be as real and as readily and directly perceptible as the motions which they produce. But it is otherwise when we turn to the motions of the stars. Here the forces have never been the objects of direct perception; all our previous experiences relate only to the apparent position of the stars. Nor do we expect in future to perceive the forces. The future experiences which we anticipate again relate only to the position of these luminous points in the heavens. It is only in the deduction of future experiences from the past that the forces of gravitation enter as transitory aids in the calculation, and then disappear from consideration. Precisely the same is true of the discussion of molecular forces, of chemical actions, and of many electric and magnetic actions. And if after more mature experience we return to the simple forces, whose existence we never doubted, we learn that these forces which we had perceived with convincing certainty, were after all not real. More mature mechanics tells us that what we believed to be simply the tendency of a body towards the earth, is not really such: it is the result, imagined only as a single force, of an inconceivable number of actual forces which attract the atoms of the body towards all the atoms of the universe. Here again the actual forces have never been the objects of previous experience; nor do we expect to come across them in future experiences. Only during the process of deducing future experiences from the past do they glide quietly in and out. But even if the forces have only been introduced by ourselves into nature, we should not on that account regard their introduction as inappropriate. We have felt sure from the beginning that unessential relations could not be altogether avoided in our images. All that we can ask is that these relations should, as far as possible, be restricted, and that a wise discretion should be observed in their use. But has physics always been sparing in the use of such relations? Has it not rather been compelled to fill the world to overflowing with forces of the most various kinds—with forces which never appeared in the phenomena, even with forces which only came into action in exceptional cases? We see a piece of iron resting upon a table, and we accordingly imagine that no causes of motion—no forces—are there present. Physics, which is based upon the mechanics considered here and necessarily determined by this basis, teaches us otherwise. Through the force of gravitation every atom of the iron is attracted by every other atom in the universe. But every atom of the iron is magnetic, and is thus connected by fresh forces with every other magnetic atom in the universe. Again, bodies in the universe contain electricity in motion, and this latter exerts further complicated forces which attract every atom of the iron. In so far as the parts of the iron themselves contain electricity, we have fresh forces to take into consideration; and in addition to these again various kinds of molecular forces. Some of these forces are not small: if only a part of these forces were effective, this part would suffice to tear the iron to pieces. But, in fact, all the forces are so adjusted amongst each other that the effect of the whole lot is zero; that in spite of a thousand existing causes of motion, no motion takes place; that the iron remains at rest. Now if we place these conceptions before unprejudiced persons, who will believe us? Whom shall we convince that we are speaking of actual things, not images of a riotous imagination? And it is for us to reflect whether we have really depicted the state of rest of the iron and its particles in a simple manner. Whether complications can be entirely avoided is questionable; but there can be no question that a system of mechanics which does avoid or exclude them is simpler, and in this sense more appropriate, than the one here considered; for this latter not only permits such conceptions, but directly obtrudes them upon us.

Let us now collect together as briefly as possible the doubts which have occurred to us in considering the customary mode of representing the principles of mechanics. As far as the form is concerned, we consider that the logical value of the separate statements is not defined with sufficient clearness. As far as the facts are concerned, it appears to us that the motions considered in mechanics do not exactly coincide with the natural motions under consideration. Many properties of the natural motions are not attended to in mechanics; many relations which are considered in mechanics are probably absent in nature. Even if these objections are acknowledged to be well founded, they should not lead us to imagine that the customary representation of mechanics is on that account either bound to or likely to lose its value and its privileged position; but they sufficiently justify us in looking out for other representations less liable to censure in these respects, and more closely conformable to the things which have to be represented.

II

There is a second image of mechanical processes which is of much more recent origin than the first. Its development from, and side by side with, the latter is closely connected with advances which physical science has made during the past few decades. Up to the middle of this century its ultimate aim was apparently to explain natural phenomena by tracing them back to innumerable actions-at-a-distance between the atoms of matter. This mode of conception corresponded completely to what we have spoken of as the first system of mechanical principles: each of the two was conditioned by the other. Now, towards the end of the century, physics has shown a preference for a different mode of thought. Influenced by the overpowering impression made by the discovery of the principle of the conservation of energy, it likes to treat the phenomena which occur in its domain as transformations of energy into new forms, and to regard as its ultimate aim the tracing back of the phenomena to the laws of the transformation of energy. This mode of treatment can also be applied from the beginning to the elementary phenomena of motion. There thus arises a new and different representation of mechanics, in which from the start the idea of force retires in favour of the idea of energy. It is this new image of the elementary processes of motion which we shall denote as the second; and to it we shall now devote our attention. In discussing the first image we had the advantage of being able to assume that it stood out plainly before the eyes of all physicists. With the second image this is not the case. It has never yet been portrayed in all its details. So far as I know,

there is no text-book of mechanics which from the start teaches the subject from the standpoint of energy, and introduces the idea of energy before the idea of force. Perhaps there has never yet been a lecture on mechanics prepared according to this plan. But to the founders of the theory of energy it was evident that such a plan was possible; the remark has often been made that in this way the idea of force with its attendant difficulties could be avoided; and in special scientific applications chains of reasoning frequently occur which belong entirely to this mode of thought. Hence we can very well sketch the rough outlines of the image; we can give the general plan according to which such a representation of mechanics must be arranged. We here start, as in the case of the first image, from four independent fundamental ideas; and the relations of these to each other will form the contents of mechanics. Two of them—space and time—have a mathematical character; the other two—mass and energy—are introduced as physical entities which are present in given quantity, and cannot be destroyed or increased. In addition to explaining these matters, it will, of course, also be necessary to indicate clearly by what concrete experiences we ultimately establish the presence of mass and energy. We here assume this to be possible and to be done. It is obvious that the amount of energy connected with given masses depends upon the state of these masses. But it is as a general experience that we must first lay down that the energy present can always be split up into two parts, of which the one is determined solely by the relative positions of the masses, while the other depends upon their absolute velocities. The first part is defined as potential energy, the second as kinetic energy. The form of the dependence of kinetic energy upon the velocity of the moving bodies is in all cases the same, and is known. The form of the dependence of potential energy upon the position of the bodies cannot be generally stated; it rather constitutes the special nature and characteristic peculiarity of the masses under consideration. It is the problem of physics to ascertain from previous experience this form for the bodies which surround us in nature. Up to this point there come essentially into consideration only three elements—space, mass, energy, considered in relation to each other. In order to settle the relations of all the four fundamental ideas, and thereby the course in time of the phenomena, we make use of one of the integral principles of ordinary mechanics which involve in their statement the idea of energy. It is not of much importance which of these we select; we can and shall choose Hamilton's principle. We thus lay down as the sole fundamental law of mechanics, in accordance with experience, the proposition that every system of natural bodies moves just as if it were assigned the problem of attaining given positions in given times, and in such a manner that the average over the whole time of the difference between kinetic and potential energy shall be as small as possible. Although this law may not be simple in form, it nevertheless represents without ambiguity the transformations of energy, and enables us to predetermine completely the course of actual phenomena for the future. In stating this new law we lay down the last of the indispensable foundations of mechanics. All that we can further add are only mathematical deductions and certain simplifications of notation which, although expedient, are not necessary. Among these latter is the idea of force, which does not enter into the foundations. Its introduction is expedient when we are considering not only masses which are connected with constant quantities of energy, but also masses which give up energy to other masses or receive it from them. Still, it is not by any new experience that it is introduced, but by a definition which can be formed in more than one way. And accordingly the properties of the force so defined are not to be ascertained by experience, but are to be deduced from the definition and the fundamental laws. Even the confirmation of these properties by experience is superfluous, unless we doubt the correctness of the whole system. Hence the idea of force as such cannot in this system involve any logical difficulties: nor can it come in question in estimating the correctness of the system; it can only increase or diminish its appropriateness.

Somewhat after the manner indicated would the principles of mechanics have to be arranged in order to adapt them to the conception of energy. The question now is, whether this second image is preferable to the first. Let us therefore consider its advantages and disadvantages.

appropriateness, since it is in this respect that the improvement is most obvious. For, to begin with, our second image of natural motions is decidedly more distinct: it shows more of their peculiarities than the first does. When we wish to deduce Hamilton's principle from the general foundations of mechanics we have to add to the latter certain assumptions as to the acting forces and the character of contingent fixed connections. These assumptions are of the most general nature, but they indicate a corresponding number of important limitations of the motions represented by the principle. And, conversely, we can deduce from the principle a whole series of relations, especially of mutual relations between every kind of possible force, which are wanting in the principles of the first image; in the second image they are present, and likewise occur, which is the important point, in nature. To prove this is the object of the papers published by von Helmholtz under the title, *Ueber die physikalische Bedeutung des Prinzips der kleinsten Wirkung*. It would be more correct to say that the fact which has to be proved forms the discovery which is demonstrated and communicated in that paper. For it is truly a discovery to find that from such general assumptions, conclusions so distinct, so weighty, and so just can be drawn. We may then appeal to that paper for confirmation of our statement; and, inasmuch as it represents the furthest advance of physics at the present time, we may spare ourselves the question whether it be possible to conform yet more closely to nature, say by limiting the permissible forms of potential energy. We shall simply emphasise this, that in respect of simplicity as well, our present image avoids the stumbling-blocks which endangered the appropriateness of the first. For if we ask ourselves the real reasons why physics at the present time prefers to express itself in terms of energy, our answer will be, Because in this way it best avoids talking about things of which it knows very little, and which do not at all affect the essential statements under consideration. We have already had occasion to remark that in tracing back phenomena to force we are compelled to turn our attention continually to atoms and molecules. It is true that we are now convinced in certain cases. But the form of the atoms, their connection, their motion in most cases—all these are entirely hidden from us; their number is in all cases immeasurably great. So that although our conception of atoms is in itself an important and interesting object for further investigation, it is in no wise specially fit to serve as a known and secure foundation for mathematical theories. To an investigator like Gustav Kirchhoff, who was accustomed to rigid reasoning, it almost gave pain to see atoms and their vibrations wilfully stuck in the middle of a theoretical deduction. The arbitrarily assumed properties of the atoms may not affect the final result. The result may be correct. Nevertheless the details of the deduction are in great part presumably false; the deduction is only in appearance a proof. The earlier mode of thought in physics scarcely allowed any choice or any way of escape. Herein lies the advantage of the conception of energy and of our second image of mechanics: that in the hypotheses of the problems there only enter characteristics which are directly accessible to experience, parameters, or arbitrary coordinates of the bodies under consideration; that the examination proceeds with the aid of these characteristics in a finite and complete form; and that the final result can again be directly translated into tangible experience. Beyond energy itself in its few forms, no auxiliary constructions enter into consideration. Our statements can be limited to the known peculiarities of the system of bodies under consideration, and we need not conceal our ignorance of the details by arbitrary and ineffectual hypotheses. All the steps in the deduction, as well as the final result, can be defended as correct and significant. These are the merits which have endeared this method to present-day physics. They are peculiar to our second image of mechanics: in the sense in which we have used the words they are to be regarded as advantages in respect of simplicity, and hence of appropriateness.

Unfortunately we begin to be uncertain as to the value of our system when we test its correctness and its logical permissibility. The question of correctness at once gives rise to legitimate doubts. Hamilton's principle can be deduced from the accepted foundations of Newtonian mechanics; but this does not by any means guarantee an accordance with nature. We have to remember that this deduction only follows if certain assumptions hold good; and also that our system claims not only to describe certain natural motions correctly, but to embrace all natural motions. We must therefore investigate whether these special assumptions which are made in addition to Newton's laws are universally true; and a single example from nature to the contrary would invalidate the correctness of our system as such, although it would not disturb in the least the validity of Hamilton's principle as a general proposition. The doubt is not so much whether our system includes the whole manifold[1] of forces, as whether it embraces the whole manifold of rigid connections which may arise between the bodies of nature. The application of Hamilton's principle to a material system does not exclude the existence of fixed connections between the chosen coordinates. But at any rate it requires that these connections be mathematically expressible by finite equations between the coordinates: it does not permit the occurrence of connections which can only be represented by differential equations. But nature itself does not appear to entirely exclude connections of this kind. They arise, for example, when bodies of three dimensions roll on one another without slipping. By such a connection, examples of which frequently occur, the position of the two bodies with respect to each other is only limited by the condition that they must always have one point of their surfaces common; but the freedom of motion of the bodies is further diminished by a degree. From the connection, then, there can be deduced more equations between the changes of the coordinates than between the coordinates themselves; hence there must amongst these equations be at least one non-integrable differential equation. Now Hamilton's principle cannot be applied to such a case; or, to speak more correctly, the application, which is mathematically possible, leads to results which are physically false. Let us restrict our consideration to the case of a sphere rolling without slipping upon a horizontal plane under the influence of its inertia alone. It is not difficult to see, without calculation, what motions the sphere can actually execute. We can also see what motions would correspond to Hamilton's principle; these would have to take place

[1] [*Mannigfaltigkeit* is thus rendered throughout.—*Tr.*]

in such a way that with constant *vis viva* the sphere would attain given positions in the shortest possible time. We can thus convince ourselves, without calculation, that the two kinds of motions exhibit very different characteristics. If we choose any initial and final positions of the sphere, it is clear that there is always one definite motion from the one to the other for which the time of motion, *i.e.* the Hamilton's integral, is a minimum. But, as a matter of fact, a natural motion from every position to every other is not possible without the co-operation of forces, even if the choice of the initial velocity is perfectly free. And even if we choose the initial and final positions so that a natural free motion between the two is possible, this will nevertheless not be the one which corresponds to a minimum of time. For certain initial and final positions the difference can be very striking. In this case a sphere moving in accordance with the principle would decidedly have the appearance of a living thing, steering its course consciously towards a given goal, while a sphere following the law of nature would give the impression of an inanimate mass spinning steadily towards it. It would be of no use to replace Hamilton's principle by the principle of least action or by any other integral principle, for there is but a slight difference of meaning between all these principles, and in the respect here considered they are quite equivalent. Only in one way can we defend the system and preserve it from the charge of incorrectness. We must decline to admit that rigid connections of the kind referred to do actually and strictly occur in nature. We must show that all so-called rolling without slipping is really rolling with a little slipping, and is therefore a case of friction. We have to rest our case upon this—that generally friction between surfaces is one of the processes which we have not yet been able to trace back to clearly understood causes; that the forces which come into play have only been ascertained quite empirically; and hence that the whole problem is one of those which we cannot at present handle without making use of force and the roundabout methods of ordinary mechanics. This defence is not quite convincing. For rolling without slipping does not contradict either the principle of energy or any other generally accepted law of physics. The process is one which is so nearly realised in the visible world

that even integration machines are constructed on the assumption that it strictly takes place. We have scarcely any right, then, to exclude its occurrence as impossible, at any rate from the mechanics of unknown systems, such as the atoms or the parts of the ether. But even if we admit that the connections in question are only approximately realised in nature, the failure of Hamilton's principle still creates difficulties in these cases. We are bound to require of every fundamental law of our mechanical system, that when applied to approximately correct relations it should always lead to approximately correct results, not to results which are entirely false. For otherwise, since all the rigid connections which we draw from nature and introduce into the calculations correspond only approximately to the actual relations, we should get into a state of hopeless uncertainty as to which admitted of the application of the law and which not. And yet we do not wish to abandon entirely the defence which we have proposed. We should prefer to admit that the doubt is one which affects the appropriateness of the system, not its correctness, so that the disadvantages which arise from it may be outweighed by other advantages.

The real difficulties first meet us when we try to arrange the elements of the system in strict accordance with the requirements of logical permissibility. In introducing the idea of energy we cannot proceed in the usual way, starting with force, and proceeding from this to force-functions, to potential energy, and to energy in general. Such an arrangement would belong to the first representation of mechanics. Without assuming any previous consideration of mechanics, we have to specify by what simple, direct experiences we propose to define the presence of a store of energy, and the determination of its amount. In what precedes we have only assumed, not shown, that such a determination is possible. At the present time many distinguished physicists tend so much to attribute to energy the properties of a substance as to assume that every smallest portion of it is associated at every instant with a given place in space, and that through all the changes of place and all the transformations of the energy into new forms it retains its identity. These physicists must have the conviction that definitions of the required kind can be found; and it is therefore permissible to assume that such definitions can be given. But when we try to throw them into a concrete form, satisfactory to ourselves and likely to command general acceptance, we become perplexed. This mode of conception as a whole does not yet seem to have arrived at a satisfactory and conclusive result. At the very beginning there arises a special difficulty, from the circumstance that energy, which is alleged to resemble a substance, occurs in two such totally dissimilar forms as kinetic and potential energy. Kinetic energy itself does not really require any new fundamental determination, for it can be deduced from the ideas of velocity and mass; on the other hand potential energy, which does require to be settled independently, does not lend itself at all well to any definition which ascribes to it the properties of a substance. The amount of a substance is necessarily a positive quantity; but we never hesitate in assuming the potential energy contained in a system to be negative. When the amount of a substance is represented by an analytical expression, an additive constant in the expression has the same importance as the rest; but in the expression for the potential energy of a system an additive constant never has any meaning. Lastly, the amount of any substance contained in a physical system can only depend upon the state of the system itself; but the amount of potential energy contained in given matter depends upon the presence of distant masses which perhaps have never had any influence upon the system. If the universe, and therefore the number of such distant masses, is infinite, then the amount of many forms of potential energy contained in even finite quantities of matter is infinitely great. All these are difficulties which must be removed or avoided by the desired definition of energy. We do not assert that such a definition is impossible, but as yet we cannot say that it has been framed. The most prudent thing to do will be to regard it for the present as an open question, whether the system can be developed in logically unexceptionable form.

It may be worth while discussing here whether there is any justification for another objection which might be raised as to the permissibility of this second system. In order that an image of certain external things may in our sense be permissible, not only must its characteristics be consistent amongst themselves, but they must not contradict the characteristics of other images already established in our knowledge. On the strength of this it may be said to be inconceivable that Hamilton's principle, or any similar proposition, should really play the part of a fundamental law of mechanics, and be a fundamental law of nature. For the first thing that is to be expected of a fundamental law is simplicity and plainness, whereas Hamilton's principle, when we come to look into it, proves to be an exceedingly complicated statement. Not only does it make the present motion dependent upon consequences which can only exhibit themselves in the future, thereby attributing intentions to inanimate nature; but, what is much worse, it attributes to nature intentions which are void of meaning. For the integral, whose minimum is required by Hamilton's principle, has no simple physical meaning; and for nature it is an unintelligible aim to make a mathematical expression a minimum, or to bring its variation to zero. The usual answer, which physics nowadays keeps ready for such attacks, is that these considerations are based upon metaphysical assumptions; that physics has renounced these, and no longer recognises it as its duty to meet the demands of metaphysics. It no longer attaches weight to the reasons which used to be urged from the metaphysical side in favour of principles which indicate design in nature, and thus it cannot lend ear to objections of a metaphysical character against these same principles. If we had to decide upon such a matter we should not think it unfair to place ourselves rather on the side of the attack than of the defence. A doubt which makes an impression on our mind cannot be removed by calling it metaphysical; every thoughtful mind as such has needs which scientific men are accustomed to denote as metaphysical. Moreover, in the case in question, as indeed in all others, it is possible to show what are the sound and just sources of our needs. It is true we cannot *a priori* demand from nature simplicity, nor can we judge what in her opinion is simple. But with regard to images of our own creation we can lay down requirements. We are justified in deciding that if our images are well adapted to the things, the actual relations of the things must be represented by simple relations between the images.

And if the actual relations between the things can only be represented by complicated relations, which are not even intelligible to an unprepared mind, we decide that these images are not sufficiently well adapted to the things. Hence our requirement of simplicity does not apply to nature, but to the images thereof which we fashion; and our repugnance to a complicated statement as a fundamental law only expresses the conviction that, if the contents of the statement are correct and comprehensive, it can be stated in a simpler form by a more suitable choice of the fundamental conceptions. The same conviction finds expression in the desire we feel to penetrate from the external acquaintance with such a law to the deeper and real meaning which we are convinced it possesses. If this conception is correct, the objection brought forward does really justify a doubt as to the system; but it does not apply so much to its permissibility as to its appropriateness, and comes under consideration in deciding as to the latter. However, we need not return to the consideration of this.

If we once more glance over the merits which we were able to claim for this second image, we come to the conclusion that as a whole it is not quite satisfactory. Although the whole tendency of recent physics moves us to place the idea of energy in the foreground, and to use it as the corner-stone of our structure, it yet remains doubtful whether in so doing we can avoid the harshness and ruggedness which were so disagreeable in the first image. In fact I have discussed this second mode of representation at some length, not in order to urge its adoption, but rather to show why, after due trial, I have felt obliged to abandon it.

III

A third arrangement of the principles of mechanics is that which will be explained at length in this book. Its principal characteristics will be at once stated, so that it may be criticised in the same way as the other two. It differs from them in this important respect, that it only starts with three independent fundamental conceptions, namely, those of time, space, and mass. The problem which it has to solve is to

represent the natural relations between these three, and between these three alone. The difficulties have hitherto been met with in connection with a fourth idea, such as the idea of force or of energy; this, as an independent fundamental conception, is here avoided. G. Kirchhoff has already made the remark in his *Text-book of Mechanics* that three independent conceptions are necessary and sufficient for the development of mechanics. Of course the deficiency in the manifold which thus results in the fundamental conceptions necessarily requires some complement. In our representation we endeavour to fill up the gap which occurs by the use of an hypothesis, which is not stated here for the first time; but it is not usual to introduce it in the very elements of mechanics. The nature of the hypothesis may be explained as follows.

If we try to understand the motions of bodies around us, and to refer them to simple and clear rules, paying attention only to what can be directly observed, our attempt will in general fail. We soon become aware that the totality of things visible and tangible do not form an universe conformable to law, in which the same results always follow from the same conditions. We become convinced that the manifold of the actual universe must be greater than the manifold of the universe which is directly revealed to us by our senses. If we wish to obtain an image of the universe which shall be well-rounded, complete, and conformable to law, we have to presuppose, behind the things which we see, other, invisible things—to imagine confederates concealed beyond the limits of our senses. These deep-lying influences we recognised in the first two representations; we imagined them to be entities of a special and peculiar kind, and so, in order to represent them in our image, we created the ideas of force and energy. But another way lies open to us. We may admit that there is a hidden something at work, and yet deny that this something belongs to a special category. We are free to assume that this hidden something is nought else than motion and mass again,—motion and mass which differ from the visible ones not in themselves but in relation to us and to our usual means of perception. Now this mode of conception is just our hypothesis. We assume that it is possible to conjoin with the visible masses of the universe other masses obeying the same laws, and of such a kind that the whole thereby becomes intelligible and conformable to law. We assume this to be possible everywhere and in all cases, and that there are no causes whatever of the phenomena other than those hereby admitted. What we are accustomed to denote as force and as energy now become nothing more than an action of mass and motion, but not necessarily of mass and motion recognisable by our coarse senses. Such explanations of force from processes of motion are usually called dynamical; and we have every reason for saying that physics at the present day regards such explanations with great favour. The forces connected with heat have been traced back with certainty to the concealed motions of tangible masses. Through Maxwell's labours the supposition that electro-magnetic forces are due to the motion of concealed masses has become almost a conviction. Lord Kelvin gives a prominent place to dynamical explanations of force; in his theory of vortex atoms he has endeavoured to present an image of the universe in accordance with this conception. In his investigation of cyclical systems von Helmholtz has treated the most important form of concealed motion fully, and in a manner that admits of general application; through him "concealed mass" and "concealed motion" have become current as technical expressions in German.[1] But if this hypothesis is capable of gradually eliminating the mysterious forces from mechanics, it can also entirely prevent their entering into mechanics. And if its use for the former purpose is in accordance with present tendencies of physics, the same must hold good of its use for the latter purpose. This is the leading thought from which we start. By following it out we arrive at the third image, the general outlines of which will now be sketched.

We first introduce the three independent fundamental ideas of time, space, and mass as objects of experience; and we specify the concrete sensible experiences by which time, mass, and space are to be determined. With regard to the masses we stipulate that, in addition to the masses recog-

introduced. We next bring together the relations which always obtain between these concrete experiences, and which we have to retain as the essential relations between the fundamental ideas. To begin with, we naturally connect the fundamental ideas in pairs. Relations between space and time alone form the subject of kinematics. There exists no connection between mass and time alone. Experience teaches us that between mass and space there exists a series of important relations. For we find certain purely spacial connections between the masses of nature: from the very beginning onwards through all time, and therefore independently of time, certain positions and certain changes of position are prescribed and associated as possible for these masses, and all others as impossible. Respecting these connections we can also assert generally that they only apply to the relative position of the masses amongst themselves; and further that they satisfy certain conditions of continuity, which find their mathematical expression in the fact that the connections themselves can always be represented by homogeneous linear equations between the first differentials of the magnitudes by which the positions of the masses are denoted. To investigate in detail the connections of definite material systems is not the business of mechanics, but of experimental physics: the distinguishing characteristics which differentiate the various material systems of nature from each other are, according to our conception, simply and solely the connections of their masses. Up to this point we have only considered the connections of the fundamental ideas in pairs: we now address ourselves to mechanics in the stricter sense, in which all three have to be considered together. We find that their general connection, in accordance with experience, can be epitomised in a single fundamental law, which exhibits a close analogy with the usual law of inertia. In accordance with the mode of expression which we shall use, it can be represented by the statement:—Every natural motion of an independent material system consists herein, that the system follows with uniform velocity one of its straightest paths. Of course this statement only becomes intelligible when we have given the necessary explanation of the mathematical single statement the usual law of inertia and Gauss's Principle of Least Constraint. It therefore asserts that if the connections of the system could be momentarily destroyed, its masses would become dispersed, moving in straight lines with uniform velocity; but that as this is impossible, they tend as nearly as possible to such a motion. In our image this fundamental law is the first proposition derived from experience in mechanics proper: it is also the last. From it, together with the admitted hypothesis of concealed masses and the normal connections, we can derive all the rest of mechanics by purely deductive reasoning. Around it we group the remaining general principles, according to their relations to it and to each other, as corollaries or as partial statements. We endeavour to show that the contents of mechanics, when arranged in this way, do not become less rich or manifold than its contents when it starts with four fundamental conceptions; at any rate not less rich or manifold than is required for the representation of nature. We soon find it convenient to introduce into our system the idea of force. However, it is not as something independent of us and apart from us that force now makes its appearance, but as a mathematical aid whose properties are entirely in our power. It cannot, therefore, in itself have anything mysterious to us. Thus according to our fundamental law, whenever two bodies belong to the same system, the motion of the one is determined by that of the other. The idea of force now comes in as follows. For assignable reasons we find it convenient to divide the determination of the one motion by the other into two steps. We thus say that the motion of the first body determines a force, and that this force then determines the motion of the second body. In this way force can with equal justice be regarded as being always a cause of motion, and at the same time a consequence of motion. Strictly speaking, it is a middle term conceived only between two motions. According to this conception the general properties of force must clearly follow as a necessary consequence of thought from the fundamental law; and if in possible experiences we see these properties confirmed, we can in no sense feel surprised, unless we are sceptical as to our fundamental law. Precisely the same is true of the idea of energy and of any other aids that may be introduced.

What has hitherto been stated relates to the physical content of the image, and nothing further need be said with regard to this; but it will be convenient to give here a brief explanation of the special mathematical form in which it will be represented. The physical content is quite independent of the mathematical form, and as the content differs from what is customary, it is perhaps not quite judicious to present it in a form which is itself unusual. But the form as well as the content only differ slightly from such as are familiar; and moreover they are so suited that they mutually assist one another. The essential characteristic of the terminology used consists in this, that instead of always starting from single points, it from the beginning conceives and considers whole systems of points. Every one is familiar with the expressions "position of a system of points," and "motion of a system of points." There is nothing unnatural in continuing this mode of expression, and denoting the aggregate of the positions traversed by a system in motion as its path. Every smallest part of this path is then a path-element. Of two path-elements one can be a part of the other: they then differ in magnitude and only in magnitude. But two path-elements which start from the same position may belong to different paths. In this case neither of the two forms part of the other: they differ in other respects than that of magnitude, and thus we say that they have different directions. It is true that these statements do not suffice to determine without ambiguity the characteristics of "magnitude" and "direction" for the motion of a system. But we can complete our definitions geometrically or analytically so that their consequences shall neither contradict themselves nor the statements we have made; and so that the magnitudes thus defined in the geometry of the system shall exactly correspond to the magnitudes which are denoted by the same names in the geometry of the point,—with which, indeed, they always coincide when the system is reduced to a point. Having determined the characteristics of magnitude and direction, we next call the path of a system straight if all its elements have the same direction, and curved if the direction of the elements changes from position to position. As in the geometry of the point, we measure curvature by the rate of variation of the direction with position. From these definitions we at once get a whole series of relations; and the number of these increases as soon as the freedom of motion of the system under consideration is limited by its connections. Certain classes of paths which are distinguished among the possible ones by peculiar simple properties then claim special attention. Of these the most important are those paths which at each of their positions have the least possible curvature: these we shall denote as the straightest paths of the system. These are the paths which are referred to in the fundamental law, and which have already been mentioned in stating it. Another important type consists of those paths which form the shortest connection between any two of their positions: these we shall denote as the shortest paths of the system. Under certain conditions the ideas of straightest and shortest paths coincide. The relation is perfectly familiar in connection with the theory of curved surfaces; nevertheless it does not hold good in general and under all circumstances. The compilation and arrangement of all the relations which arise here belong to the geometry of systems of points. The development of this geometry has a peculiar mathematical attraction; but we only pursue it as far as is required for the immediate purpose of applying it to physics. A system of n points presents a $3n$-manifold of motion,—although this may be reduced to any arbitrary number by the connections of the system. Hence there arise many analogies with the geometry of space of many dimensions; and these in part extend so far that the same propositions and notations can apply to both. But we must note that these analogies are only formal, and that, although they occasionally have an unusual appearance, our considerations refer without exception to concrete images of space as perceived by our senses. Hence all our statements represent possible experiences; if necessary, they could be confirmed by direct experiments, viz. by measurements made with models. Thus we need not fear the objection that in building up a science dependent upon experience, we have gone outside the world of experience. On the other hand, we are bound to answer the question how a new, unusual, and comprehensive mode of expression justifies itself; and what advantages we expect from using it. In answering this question we specify as the first advantage that it enables us to render the most general

and comprehensive statements with great simplicity and brevity. In fact, propositions relating to whole systems do not require more words or more ideas than are usually employed in referring to a single point. Here the mechanics of a material system no longer appears as the expansion and complication of the mechanics of a single point; the latter, indeed, does not need independent investigation, or it only appears occasionally as a simplification and a special case. If it is urged that this simplicity is only artificial, we reply that in no other way can simple relations be secured than by artificial and well-considered adaptation of our ideas to the relations which have to be represented. But in this objection there may be involved the imputation that the mode of expression is not only artificial, but far-fetched and unnatural. To this we reply that there may be some justification for regarding the consideration of whole systems as being more natural and obvious than the consideration of single points. For, in reality, the material particle is simply an abstraction, whereas the material system is presented directly to us. All actual experience is obtained directly from systems; and it is only by processes of reasoning that we deduce conclusions as to possible experiences with single points. As a second merit, although not a very important one, we specify the advantage of the form in which our mathematical mode of expression enables us to state the fundamental law. Without this we should have to split it up into Newton's first law and Gauss's principle of least constraint. Both of these together would represent accurately the same facts; but in addition to these facts they would by implication contain something more, and this something more would be too much. In the first place they suggest the conception, which is foreign to our system of mechanics, that the connections of the material system might be destroyed; whereas we have denoted them as being permanent and indestructible throughout. In the second place we cannot, in using Gauss's principle, avoid suggesting the idea that we are not only stating a fact, but also the cause of this fact. We cannot assert that nature always keeps a certain quantity, which we call constraint, as small as possible, without suggesting that this quantity signifies something which is for nature itself a constraint,—an uncomfortable feeling. We cannot assert that nature acts like a judicious calculator reducing his observations, without suggesting that deliberate intention underlies the action. There is undoubtedly a special charm in such suggestions; and Gauss felt a natural delight in giving prominence to it in his beautiful discovery, which is of fundamental importance in our mechanics. Still, it must be confessed that the charm is that of mystery; we do not really believe that we can solve the enigma of the world by such half-suppressed allusions. Our own fundamental law entirely avoids any such suggestions. It exactly follows the form of the customary law of inertia, and like this it simply states a bare fact without any pretence of establishing it. And as it thereby becomes plain and unvarnished, in the same degree does it become more honest and truthful. Perhaps I am prejudiced in favour of the slight modification which I have made in Gauss's principle, and see in it advantages which will not be manifest to others. But I feel sure of general assent when I state as the third advantage of our method, that it throws a bright light upon Hamilton's method of treating mechanical problems by the aid of characteristic functions. During the sixty years since its discovery this mode of treatment has been well appreciated and much praised; but it has been regarded and treated more as a new branch of mechanics, and as if its growth and development had to proceed in its own way and independently of the usual mechanics. In our form of the mathematical representation, Hamilton's method, instead of having the character of a side branch, appears as the direct, natural, and, if one may so say, self-evident continuation of the elementary statements in all cases to which it is applicable. Further, our mode of representation gives prominence to this: that Hamilton's mode of treatment is not based, as is usually assumed, on the special physical foundations of mechanics; but that it is fundamentally a purely geometrical method, which can be established and developed quite independently of mechanics, and which has no closer connection with mechanics than any other of the geometrical methods employed in it. It has long since been remarked by mathematicians that Hamilton's method contains purely geometrical truths, and that a peculiar mode of expression, suitable to it, is required in order to express these clearly. But this fact has only come to light in a somewhat perplexing form, namely, in the analogies between ordinary mechanics and the geometry of space of many dimensions, which have been discovered by following out Hamilton's thoughts. Our mode of expression gives a simple and intelligible explanation of these analogies. It allows us to take advantage of them, and at the same time it avoids the unnatural admixture of supra-sensible abstractions with a branch of physics.

We have now sketched the content and form of our third image as far as can be done without trenching upon the contents of the book; far enough to enable us to submit it to criticism in respect of its permissibility, its correctness, and its appropriateness. I think that as far as logical permissibility is concerned it will be found to satisfy the most rigid requirements, and I trust that others will be of the same opinion. This merit of the representation I consider to be of the greatest importance, indeed of unique importance. Whether the image is more appropriate than another; whether it is capable of including all future experience; even whether it only embraces all present experience, all this I regard almost as nothing compared with the question whether it is in itself conclusive, pure and free from contradiction. For I have not attempted this task because mechanics has shown signs of inappropriateness in its applications, nor because it in any way conflicts with experience, but solely in order to rid myself of the oppressive feeling that to me its elements were not free from things obscure and unintelligible. What I have sought is not the only image of mechanics, nor yet the best image; I have only sought to find an intelligible image and to show by an example that this is possible and what it must look like. We cannot attain to perfection in any direction; and I must confess that, in spite of the pains I have taken with it, the image is not so convincingly clear but that in some points it may be exposed to doubt or may require defence. And yet it seems to me that of objections of a general nature there is only a single one which is so pertinent that it is worth while to anticipate and remove it. It relates to the nature of the rigid connections which we assume to exist between the masses, and which are absolutely indispensable in our system. Many physicists will at first be of opinion that by means of these connections forces are introduced into the elements of mechanics, and are introduced in a way which is secret, and therefore not permissible. For, they will assert, rigid connections are not conceivable without forces; they cannot come into existence except by the action of forces. To this we reply—Your assertion is correct for the mode of thought of ordinary mechanics, but it is not correct independently of this mode of thought; it does not carry conviction to a mind which considers the facts without prejudice and as if for the first time. Suppose we find in any way that the distance between two material particles remains constant at all times and under all circumstances. We can express this fact without making use of any other conceptions than those of space; and the value of the fact stated, as a fact, for the purpose of foreseeing future experience and for all other purposes, will be independent of any explanation of it which we may or may not possess. In no case will the value of the fact be increased, or our understanding of it improved, by putting it in the form—"Between these masses there acts a force which keeps them at a constant distance from one another," or "Between them there acts a force which makes it impossible for their distance to alter from its fixed value." But it will be urged that this latter explanation, although apparently only a ludicrous circumlocution, is nevertheless correct. For all the connections of the actual world are only approximately rigid; and the appearance of rigidity is only produced by the action of the elastic forces which continually annul the small deviations from the position of rest. To this we reply as follows:—With regard to rigid connections which are only approximately realised, our mechanics will naturally only state as a fact that they are approximately satisfied; and for the purpose of this statement the idea of force is not required. If we wish to proceed to a second approximation and to take into consideration the deviations, and with them the elastic forces, we shall make use of a dynamical explanation for these as for all forces. In seeking the actual rigid connections we shall perhaps have to descend to the world of atoms. But such considerations are out of place here; they do not affect the question whether it is logically permissible to treat of fixed connections as independent of forces and precedent to them. All that I wished to show was that this question must be answered in the affirmative, and this I believe I have done. This being so, we can deduce the properties and behaviour of the forces from the nature of the fixed connections without being guilty of a *petitio principii*. Other objections of a similar kind are possible, but I believe they can be removed in much the same way.

By way of giving expression to my desire to prove the logical purity of the system in all its details, I have thrown the representation into the older synthetic form. For this purpose the form used has the merit of compelling us to specify beforehand, definitely even if monotonously, the logical value which every important statement is intended to have. This makes it impossible to use the convenient reservations and ambiguities into which we are enticed by the wealth of combinations in ordinary speech. But the most important advantage of the form chosen is that it is always based upon what has already been proved, never upon what is to be proved later on: thus we are always sure of the whole chain if we sufficiently test each link as we proceed. In this respect I have endeavoured to carry out fully the obligations imposed by this mode of representation. At the same time it is obvious that the form by itself is no guarantee against error or oversight; and I hope that any chance defects will not be the more harshly criticised on account of the somewhat presumptuous mode of presentation. I trust that any such defects will be capable of improvement and will not affect any important point. Now and again, in order to avoid excessive prolixity, I have consciously abandoned to some extent the rigid strictness which this mode of representation properly requires. Before proceeding to mechanics proper, as dependent upon physical experience, I have naturally discussed those relations which follow simply and necessarily from the definitions adopted and from mathematics; the connection of these latter with experience, if any, is of a different nature from that of the former. Moreover, there is no reason why the reader should not begin with the second book. The matter with which he is already familiar and the clear analogy with the dynamics of a particle will enable him easily to guess the purport of the propositions in the first book. If he admits the appropriateness of the mode of expression used, he can at any time return to the first book to convince himself of its permissibility.

We next turn to the second essential requirement which our image must satisfy. In the first place there is no doubt that the system correctly represents a very large number of natural motions. But this does not go far enough; the system must include all natural motions without exception. I think that this, too, can be asserted of it; at any rate in the sense that no definite phenomena can at present be mentioned which would be inconsistent with the system. We must of course admit that we cannot extend a rigid examination to all phenomena. Hence the system goes a little beyond the results of assured experience; it therefore has the character of a hypothesis which is accepted tentatively and awaits sudden refutation by a single example or gradual confirmation by a large number of examples. There are in especial two places in which we go beyond assured experience: firstly, in our limitation of the possible connections; secondly, in the dynamical explanation of force. What right have we to assert that all natural connections can be expressed by linear differential equations of the first order? With us this assumption is not a matter of secondary importance which we might do without. Our system stands or falls with it; for it raises the question whether our fundamental law is applicable to connections of the most general kind. And yet connections of a more general kind are not only conceivable, but they are permitted in ordinary mechanics without hesitation. There nothing prevents us from investigating the motion of a point where its path is only limited by the supposition that it makes a given angle with a given plane, or that its radius of curvature is always proportional to another given length. These are conditions which are not permissible in our system. But why are we certain that they are debarred by the nature of things? We might reply that these and similar connections cannot be realised by any practical mechanism; and in this respect we might appeal to the great authority of Helmholtz's name. But in every example possibilities might be overlooked; and ever so many examples would not suffice to

substantiate the general assertion. It seems to me that the reason for our conviction should more properly be stated as follows. All connections of a system which are not embraced within the limits of our mechanics, indicate in one sense or another a discontinuous succession of its possible motions. But as a matter of fact it is an experience of the most general kind that nature exhibits continuity in infinitesimals everywhere and in every sense: an experience which has crystallised into firm conviction in the old proposition—*Natura non facit saltus*. In the text I have therefore laid stress upon this: that the permissible connections are defined solely by their continuity; and that their property of being represented by equations of a definite form is only deduced from this. We cannot attain to actual certainty in this way. For this old proposition is indefinite, and we cannot be sure how far it applies—how far it is the result of actual experience, and how far the result of arbitrary assumption. Thus the most conscientious plan is to admit that our assumption as to the permissible connections is of the nature of a tentatively accepted hypothesis. The same may be said with respect to the dynamical explanation of force. We may indeed prove that certain classes of concealed motions produce forces which, like actions-at-a-distance in nature, can be represented to any desired degree of approximation as differential coefficients of force-functions. It can be shown that the form of these force-functions may be of a very general nature; and in fact we do not deduce any restrictions for them. But on the other hand it remains for us to prove that any and every form of the force-functions can be realised; and hence it remains an open question whether such a mode of explanation may not fail to account for some one of the forms occurring in nature. Here again we can only bide our time so as to see whether our assumption is refuted, or whether it acquires greater and greater probability by the absence of any such refutation. We may regard it as a good omen that many distinguished physicists tend more and more to favour the hypothesis. I may mention Lord Kelvin's theory of vortex-atoms: this presents to us an image of the material universe which is in complete accord with the principles of our mechanics. And yet our mechanics in no wise demands such great simplicity and limitation of assumptions as Lord Kelvin has imposed upon himself. We need not abandon our fundamental propositions if we were to assume that the vortices revolved about rigid or flexible, but inextensible, nuclei; and instead of assuming simply incompressibility we might subject the all-pervading medium to much more complicated conditions, the most general form of which would be a matter for further investigation. Thus there appears to be no reason why the hypothesis admitted in our mechanics should not suffice to explain the phenomena.

We must, however, make one reservation. In the text we take the natural precaution of expressly limiting the range of our mechanics to inanimate nature; how far its laws extend beyond this we leave as quite an open question. As a matter of fact we cannot assert that the internal processes of life follow the same laws as the motions of inanimate bodies; nor can we assert that they follow different laws. According to appearance and general opinion there seems to be a fundamental difference. And the same feeling which impels us to exclude from the mechanics of the inanimate world as foreign every indication of an intention, of a sensation, of pleasure and pain,—this same feeling makes us unwilling to deprive our image of the animate world of these richer and more varied conceptions. Our fundamental law, although it may suffice for representing the motion of inanimate matter, appears (at any rate that is one's first and natural impression) too simple and narrow to account for even the lowest processes of life. It seems to me that this is not a disadvantage, but rather an advantage of our law. For while it allows us to survey the whole domain of mechanics, it shows us what are the limits of this domain. By giving us only bare facts, without attributing to them any appearance of necessity, it enables us to recognise that everything might be quite different. Perhaps such considerations will be regarded as out of place here. It is not usual to treat of them in the elements of the customary representation of mechanics. But there the complete vagueness of the forces introduced leaves room for free play. There is a tacit stipulation that, if need be, later on a contrast between the forces of animate and inanimate nature may be established. In our representation the outlines

of the image are from the first so sharply delineated, that any subsequent perception of such an important division becomes almost impossible. We are therefore bound to refer to this matter at once, or to ignore it altogether.

As to the appropriateness of our third image we need not say much. In respect of distinctness and simplicity, as the contents of the book will show, we may assign to it about the same position as to the second image; and the merits to which we drew attention in the latter are also present here. But the permissible possibilities are somewhat more extensive than in the second image. For we pointed out that in the latter certain rigid connections were wanting; by our fundamental assumptions these are not excluded. And this extension is in accordance with nature, and is therefore a merit; nor does it prevent us from deducing the general properties of natural forces, in which lay the significance of the second image. The simplicity of this image, as of the second, is very apparent when we consider their physical applications. Here, too, we can confine our consideration to any characteristics of the material system which are accessible to observation. From their past changes we can deduce future ones by applying our fundamental law, without any necessity for knowing the positions of all the separate masses of the system, or for concealing our ignorance by arbitrary, ineffectual, and probably false hypotheses. But as compared with the second image, our third one exhibits simplicity also in adapting its conceptions so closely to nature that the essential relations of nature are represented by simple relations between the ideas. This is seen not only in the fundamental law, but also in its numerous general corollaries which correspond to the so-called principles of mechanics. Of course it must be admitted that this simplicity only obtains when we are dealing with systems which are completely known, and that it disappears as soon as concealed masses come in. But even in these cases the reason of the complication is perfectly obvious. The loss of simplicity is not due to nature, but to our imperfect knowledge of nature. The complications which arise are not simply a possible, but a necessary result of our special assumptions. It must also be admitted that the co-operation of concealed masses, which is the remote and special case from the standpoint of our mechanics, is the commonest case in the problems which occur in daily life and in the arts. Hence it will be well to point out again that we have only spoken of appropriateness in a special sense—in the sense of a mind which endeavours to embrace objectively the whole of our physical knowledge without considering the accidental position of man in nature, and to set forth this knowledge in a simple manner. The appropriateness of which we have spoken has no reference to practical applications or the needs of mankind. In respect of these latter it is scarcely possible that the usual representation of mechanics, which has been devised expressly for them, can ever be replaced by a more appropriate system. Our representation of mechanics bears towards the customary one somewhat the same relation that the systematic grammar of a language bears to a grammar devised for the purpose of enabling learners to become acquainted as quickly as possible with what they will require in daily life. The requirements of the two are very different, and they must differ widely in their arrangement if each is to be properly adapted to its purpose.

In conclusion, let us glance once more at the three images of mechanics which we have brought forward, and let us try to make a final and conclusive comparison between them. After what we have already said, we may leave the second image out of consideration. We shall put the first and third images on an equality with respect to permissibility, by assuming that the first image has been thrown into a form completely satisfactory from the logical point of view. This we have already assumed to be possible. We shall also put both images on an equality with respect to appropriateness, by assuming that the first image has been rendered complete by suitable additions, and that the advantages of both in different directions are of equal value. We shall then have as our sole criterion the correctness of the images: this is determined by the things themselves and does not depend on our arbitrary choice. And here it is important to observe that only one or the other of the two images can be correct: they cannot both at the same time be correct. For if we try to express as briefly as possible the essential relations of the two representations, we

come to this. The first image assumes as the final constant elements in nature the relative accelerations of the masses with reference to each other: from these it incidentally deduces approximate, but only approximate, fixed relations between their positions. The third image assumes as the strictly invariable elements of nature fixed relations between the positions: from these it deduces when the phenomena require it approximately, but only approximately, invariable relative accelerations between the masses. Now, if we could perceive natural motions with sufficient accuracy, we should at once know whether in them the relative acceleration, or the relative relations of position, or both, are only approximately invariable. We should then know which of our two assumptions is false; or whether both are false; for they cannot both be simultaneously correct. The greater simplicity is on the side of the third image. What at first induces us to decide in favour of the first is the fact that in actions-at-a-distance we can actually exhibit relative accelerations which, up to the limits of our observation, appear to be invariable; whereas all fixed connections between the positions of tangible bodies are soon and easily perceived by our senses to be only approximately constant. But the situation changes in favour of the third image as soon as a more refined knowledge shows us that the assumption of invariable distance-forces only yields a first approximation to the truth; a case which has already arisen in the sphere of electric and magnetic forces. And the balance of evidence will be entirely in favour of the third image when a second approximation to the truth can be attained by tracing back the supposed actions-at-a-distance to motions in an all-pervading medium whose smallest parts are subjected to rigid connections; a case which also seems to be nearly realised in the same sphere. This is the field in which the decisive battle between these different fundamental assumptions of mechanics must be fought out. But in order to arrive at such a decision it is first necessary to consider thoroughly the existing possibilities in all directions. To develop them in one special direction is the object of this treatise,—an object which must necessarily be attained even if we are still far from a possible decision, and even if the decision should finally prove unfavourable to the image here developed.