On the Origin and Significance[1] of the Geometrical Axioms

Lecture held in the Docentenverein of Heidelberg in the year 1870

By Hermann Helmholtz

(Translated from *Populäre Wissenschaftliche Vorträge von H. Helmholtz* [Braunschweig, 1876], Heft 3, pp. 23-51.)

The fact that there can be a science of such a character as that of geometry--that a science can be built up in such a way--has from ancient times necessarily laid the highest claims upon the attention of all who felt a concern for the fundamental questions of epistemology. Among the branches of human science there is no other that makes its appearance so: sprung forth complete, light a brazen-armed Minerva out of the head of Zeus;[2] there is none before whose annihilating ægis contradiction and doubt have so little dared to raise their eyes. With this, geometry is exempt from the wearisome tedious task of gathering facts of experience, as the natural sciences in the narrower sense have to do; instead, the exclusive form of its scientific procedure is deduction. Conclusion is developed from conclusion; and yet no one of sound sense doubts in the end that these geometrical propositions must find their very practical employment upon the reality that surrounds us. Land-surveying as well as architecture, machine-construction as well as mathematical physics, continually calculate spatial relationships of the most varied sorts on the basis of geometrical propositions; they expect the results of their constructions and experiments to conform to these calculations; and to this day, no case is known in which they have been deceived in this expectation--provided they have calculated correctly, from adequate data.

And indeed the fact that geometry exists and performs such things has always been used, in the controversy over the question that constitutes as it were the central point of all the oppositions of philosophical systems, in order to demonstrate by an imposing example that a cognition of propositions having real content is possible without any corresponding foundation drawn from experience. In particular, in the answer to *Kant's* famous question: "How are synthetic a priori propositions possible?", the geometrical axioms constitute just those examples that seem to show most evidently that synthetic a priori propositions are possible at all. Further, the circumstance that such propositions exist and compel our conviction with necessity is regarded by him as a proof that space is an a priori given form of all outer intuition. He appears thus not just to claim for that a priori given form the character of a purely formal schema, in itself contentless, in which every arbitrary content of experience would fit, but rather to include certain particularities of the schema as well, in virtue of which only a content restricted by laws in a certain way can enter into that form and become intuitable.*)

Now it is just this epistemological interest of geometry that gives me the courage to speak of geometrical things in a gathering whose members have penetrated only the smallest part further, in mathematical studies, than required by the school-curriculum. And fortunately, the geometrical

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[1][Or "meaning": Ger., *Bedeutung.]*

[2][The hybrid mythological nomenclature--Roman and Greek--is the author's. (Here and in the following, numbered footnotes are notes of the translator; those marked by single or multiple asterisk's are H.'s own.--A few of H.'s notes, referring to other essays in the collection, are omitted. Matter in brackets is supplied by the translator.)]

*) In his book "Über die Grenzen der Philosophie", Mr. *W. Tobias* maintains that earlier statements of mine to this same effect are a misconstruction of *Kant*'s view. But *Kant* specifically cites the propositions that the straight line is the shortest (Critique of Pure Reason, Introduction V, 2nd ed. p. 16); that space has three dimensions (ibid. Part I, Sect. I. §3, p. 41); that only one straight line is possible between two points (ibid. Part II, Sect. I, of the Axioms of Intuition p. 204); as propositions "which express the conditions of sensible intuition a priori". But whether these propositions are given originally in spatial intuition, or whether the latter only gives the starting-point from which the understanding can develop such propositions a priori--a question on which my critic lays weight--is of no concern here.

information ordinarily taught in the Gymnasium will suffice, as I believe, to make the propositions to be discussed in the following at least intelligible to you.

I intend, namely, to report to you on a series of recent mutually connected mathematical works that bear upon the geometrical axioms, their relations to experience, and the logical possibility of replacing them by others.

Since the original works of the mathematicians upon these matters, designed in the first instance only to furnish proofs for the experts in a field that demands a greater power of abstraction than almost any other, are rather inaccessible to the non-mathematician, I want to try to make plain,[3] for a non-mathematician as well, what these works are about. I need hardly remark that my discussion makes no claim to give a proof of the correctness of the new insights.. Whoever seeks such a proof must take the trouble to study the original works.

He who has once passed through the gates of the first elementary propositions into geometry--that is, the mathematical theory of space--finds before him on his further way that gap-free chain of arguments I have already spoken of, through which ever more manifold and complicated space-forms receive their laws. But in those first elements a few propositions are set up, of which geometry herself declares that she cannot prove them--that she must simply count upon it that everyone who understands the sense of these propositions will concede their correctness. These are the so-called axioms of geometry. To these belongs first the proposition that, if one calls the shortest line that can be drawn between two points a *straight line,* there can be between two points only one such straight line, and not two different ones. It is further an axiom that through any three points of space that do not lie in a straight line a plane can be passed--that is, a surface in which every straight line that joins two of its points is contained entirely. Another (much-discussed) axiom states that through a point lying outside a straight line there can be passed only one single line parallel to that first one, and not two different such lines--(one calls two lines parallel if they lie in one and the same plane and never intersect, no matter how far they may be prolonged). In addition, the geometrical axioms express propositions that specify the number of dimensions of space and of its surfaces, lines, and points; and that elucidate the concept of the continuity of these structures--such propositions as that the boundary of a body is a surface, that of a surface a line, that of a line a point, and that the point is indivisible; and such propositions as that by motion of a point a line is described, by motion of a line either a line or a surface, by motion of a surface either a surface or a body, but by the motion of a body always just again a body.

is this literal?

Where, then, do such propositions come from--unprovable, and yet undoubtedly correct, in the field of a science where everything else has yielded to the mastery of argument? Are they an inheritance from the god-like source of our reason, as the idealistic philosophers believe; or has the ingenuity of the generations of mathematicians that have arisen so far just not been sufficient to find the proof? Naturally, every new disciple of geometry who approaches this science with fresh ardor attempts to be the fortunate one who surpasses all predecessors. And it is quite right that each should attempt this anew: for only through the fruitlessness of his own attempts could one convince oneself, in the state of affairs heretofore, of the impossibility of the proof. Unfortunately there have also been, recurrently from time to time, individual brooders who have entangled themselves so long and so deeply in complicated trains of argument that they could no longer discover the errors committed along the way, and have believed themselves to have solved the problem. The proposition of parallels, in particular, has called forth a great number of specious proofs.

The greatest difficulty in these investigations has always consisted in this, that all too easily results of everyday life have mixed in, as apparent necessities of thought, with the logical development of concepts--so long as the only method of geometry was the method of intuition taught by *Euclides*. In particular, it is extraordinarily difficult, proceeding in this way, to be clear at all points as to whether, in the steps that one arranges one after another for the proof, one has not involuntarily and unwittingly brought to aid certain very general results of experience, which have already taught us in practice the feasibility of certain prescribed parts of the procedure. The well-schooled geometer asks of every auxiliary line that he draws for any proof whether it will always be possible to draw a line of the required sort. It is well known that construction procedures play an essential role in the system of geometry.

log. devel. of concepts? they devel. through an 'internal' logic, or through the relations to other concepts, or both?

[3][anschaulich: "intuitive" or "visualizable"]

Superficially regarded, these look like practical applications, inserted as exercises for pupils. But in truth they establish the existence of certain structures. They either show that points, straight lines, circles, of the sort the problem requires to be constructed, are possible under all conditions, or they determine the possible exceptional cases. The point on which the investigations to be discussed in the following turn is essentially of this sort. The foundation of all proofs in the *Euclid*ean method is the proof of the congruence of the relevant lines, angles, plane figures, bodies, etc. In order to make the congruence evident,[4] one imagines the geometrical structures involved to be brought together--of course without altering their form and dimensions. That this is in fact possible and feasible we have all experienced from earliest childhood. If, however, we want to build necessities of thought upon this assumption of free mobility of fixed spatial structures with unaltered form towards every part of space, then we must raise the question whether this assumption does not involve some logically undemonstrated presupposition. We shall presently see that it does in fact involve such a presupposition--and, indeed, one very rich in consequences. But if it does so, then every proof of congruence is based upon a fact taken only from experience.

I adduce these considerations in the first instance only to make clear the kind of difficulties we come upon in the complete analysis of all our presuppositions according to the method of intuition. We escape these difficulties when we apply to the investigation of principles the analytic methods developed by modern analytic geometry. The whole execution of the analysis (calculation) is a purely logical operation: it can yield no relation between the magnitudes subject to calculation that is not already contained in the equations that form the starting-point of the calculation. The recent investigations mentioned have therefore ben conducted almost exclusively by means of the purely abstract methods of analytic geometry.

Yet it is also possible, after the abstract method has taught us to know the points concerned, to give in some degree an intuition of these points; and we can do this best if we descend to a narrower domain than our own spatial world. Let us conceive--there is no impossibility in it--beings, endowed with understanding, of only two dimensions, living and moving about on the surface of some one of our solid bodies. We suppose that they do not have the ability to perceive anything outside this surface, but that they are able to have perceptions, similar to our own, within the extension of the surface in which they move. If such beings should develop a geometry for themselves, they would of course ascribe to their space only two dimensions. They would discover that a moving point describes a line and a moving line a surface--which for them would be the most complete spatial structure they know. But they would be as little able to form an image of a further spatial structure, that should arise when a surface was moved out of its superficial space, as we can do of a structure that should arise from the motion of a body out of the space we know. By the much misused expressions "to imagine" or "to be able to conceive how something happens" I understand--and I do not see how one can understand anything else by them, without sacrificing all sense of the expression--that one can portray to himself the series of sensible impressions that one would have if such a thing occurred in a particular case. If, then, no sensible impression whatever is known that would relate to such a never-observed occurrence as a motion along a fourth dimension would be for us--and, for those surface-beings, a motion along the (to us well known) third dimension of space--then such an "imagining" is not possible; just as little as a man absolutely blind from childhood will be able to "imagine" the colors, even if one could give him a conceptual description of them.

Those surface-beings would, further, be able to draw shortest lines in their superficial space. These would not necessarily be straight lines in our sense, but what in geometrical terminology we should call *geodesic lines* of the surface on which they live: lines such as a stretched thread describes that one lays upon the surface, and that can slide along it unimpeded. I shall allow myself in the sequel to designate such lines as the *straightest* lines of the designated surface (or of a given space), in order to emphasize their analogy with the straight line in the plane. I hope by this expression to bring the concept nearer to the intuition of my non-mathematical auditors, yet without giving rise to confusions.

If, then, beings of this sort lived on an infinite plane, they would set up exactly the same geometry as is contained in our own planimetry. They would assert that between two points only one straight line is

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[4][anschaulich]

possible, that through a third point outside that line only one line parallel to the first can be passed, that moreover a straight line can be prolonged into the infinite without its ends ever meeting, and so forth. Their space could be of infinite extent; but even if they should come upon limits to their motion and perception, they would be able to imagine intuitively a continuation beyond these limits, and in this image their space would appear to them of infinite extent--just as ours does to us, although we too cannot bodily leave our earth, and our sight reaches only so far as there are visible fixed stars.

But, now, intelligent beings of this sort could also live on the surface of a sphere. Their shortest or straightest line between two points would then be an arc of the great circle that passes through those points. Every great circle that goes through two given points is divided by them into two parts. If these are unequal, the smaller is, to be sure, the unique shortest line on the sphere that subsists between these two points. But the other, larger, arc of the same great circle is also a geodesic or straightest line--i.e., each fairly small piece of it is a shortest line between its own two end-points. On account of this circumstance, we cannot flatly identify the concept of the geodesic or straightest line with that of the shortest line. If, now, the two given points are end-points of a single diameter of the sphere, then all the planes that pass through this diameter cut out of the sphere semicircles, each of which is a shortest line between the two end-points. In such a case there are thus infinitely many equal shortest lines between the two given points. Consequently the axiom that only one shortest line exists between two points would not be valid for the sphere-dwellers without a certain exception.

As for parallel lines, the inhabitants of the sphere would not know them at all. They would maintain that every arbitrary pair of straightest lines, suitably prolonged, must finally intersect--not just in one but in two points. The sum of the angles in a triangle would always exceed two right angles--and the more, the greater the surface of the triangle. For just this reason, the concept of geometrical similarity of form between larger and smaller figures would be lacking; for a larger triangle must necessarily have different angles from a smaller one. Their space would have to be found, or at least imagined, to be unbounded indeed, but of finite extent.

It is clear that the beings on the sphere, with the same logical abilities as those on the plane. would yet have to set up an altogether different system of geometrical axioms from those of the latter, or from those we ourselves set up in our space of three dimensions. These examples already show us that, according to the character of their dwelling-place, different geometrical axioms would have to be set up by beings whose powers of understanding could entirely correspond with our own.

well – th 'powers of understanding' don't entirely correspond w/ our own – th only 2-dim. at all – how much does th really affe.

But let us go further. Let us conceive of intelligent beings existing on the surface of an egg-shaped body. Among any three points of such a surface one could draw shortest lines and so construct a triangle. But if one attempted to construct congruent triangles at different places of this surface, it would become evident that when two triangles have equally long sides, their angles do not turn out equal. The angle-sum of the triangle drawn at the sharper end of the egg would differ more from two right angles than if a triangle with the same sides were drawn at the blunter end; from this it follows that on such a surface not even so dimple a spatial structure as a triangle could be moved from one place to another without a change of its form. In the same way, it would be found that if circles of equal radius (the lengths of the radii always measured along shortest lines of the surface) were constructed at different places of such a surface, the circumference of the one at the blunt end would turn out greater than at the sharper end.

From this it further follows that it is a special geometrical property of a surface for figures lying in it to admit of free displacement without alteration of any of their lines and angles, and that this will not be the case on every sort of surface. The condition for a surface to possess this important property was already demonstrated by *Gauß* in his famous treatise on the curvature of surfaces.[5] It is that what he called the "measure of curvature" (namely, the reciprocal of the product of both principal radii of curvature) have the same magnitude everywhere along the whole extent of the surface.

Gauß proved at the same time that this measure of curvature does not change when the surface is bent without suffering an expansion or contraction in any part. Thus, we can roll up a plane sheet of paper into a cylinder or a cone (or horn), without any change in the measurements--taken along the surface of the paper--of its figures. And in the same way we can roll the hemispherical closed half of a pig's bladder into a spindle shape without changing the measurements in this surface. The geometry on a

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[5][Disquisitiones generales circa superficies curvas. (1828)]

plane will thus also be the same as in a cylindrical surface. We must only imagine, in the latter case, that unboundedly many layers of this surface lie, like the layers of a rolled-up sheet of paper, one over the other; and that on each entire circuit of the cylinder's circumference one comes to another layer, different from that in which one was before.

These remarks are necessary in order to be able to give you an image of a sort of surface whose geometry is on the whole similar to that of the plane, but for which the axiom of parallels does not hold. This is a sort of curved surface which stands, in geometrical respects, as it were opposite to the sphere, and which has therefore been called the *pseudospherical surface* by the distinguished Italian mathematician *E. Beltrami,**) who has investigated its properties. It is a saddle-shaped surface, of which in our space only bounded pieces or strips can be connectedly represented, but which one can yet conceive to be continued in all directions into the infinite--since one can conceive each piece that lies on the boundary of the constructed part of the surface to be shoved back into the midst of that part, and then to be continued. The displaced piece of surface must, in this process, change its flexure, but not its dimensions--just as, on a cone made by rolling a plane into a horn-shape, one can push a sheet of paper about. (Such a sheet conforms everywhere to the cone-surface; but nearer the vertex of the cone it must be bent more; and it cannot be moved over and past the vertex in such a way as to stay conformed to both the existing cone and its ideal continuation beyond the vertex.)

Like the plane and the sphere, the pseudospherical surfaces are of constant curvature, so that every piece of one of them can be applied with perfect fir to every other place on the same surface; thus all figures constructed at one place in the surface can be transferred, in perfectly congruent form and with perfect equality of all dimensions lying in the surface itself, to any other place. The measure of curvature established by *Gauß*, which is positive for the sphere and zero for the plane, would have for the pseudospherical surfaces a constant negative value--because the two principal curvatures of a saddle-shaped surface turn their concavities towards opposite sides.

A strip of a pseudospherical surface can, for example, be developed as a surface of a ring. Conceive a surface like *a a, b b,* Fig, 1, rotated about its axis of symmetry AB; the two arcs *ab* would describe such a pseudospherical surface.[6] The two edges of the surface, above at *aa* and below at *bb,* would turn outwards with ever increasing flexure, until the surface stands perpendicular to the axis--and there it would end, with infinite curvature at the edge.[7] It would also be possible to develop one-half of a pseudospherical surface in the form of a cup-shaped champagne glass with infinitely elongated, ever thinner-growing stem, as in Fig. 2. But on one side it is necessarily always bounded by a sharply ending

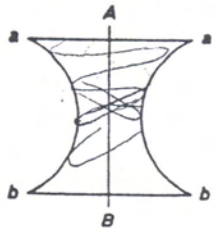

Fig. 1

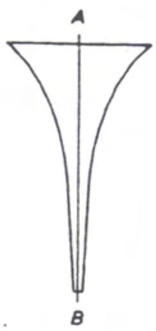

Fig. 2

*) Saggio di Interpretazione della Geometria Non-Euclidea. Napoli 1868.--Teoria fondamentale degli Spazii di Curvatura costante. Annali di Matematica. Ser. II, Tomo II, pp. 232-255.

[6][It should be noted that this depends upon a quite particular specification of the curves *ab*--not merely upon their being qualitatively "like" those in the figure.]

[7][That is, an infinite curvature of the "profile"--the arcs *ab;* the Gaussian curvature of the surface itself is, of course, constant and finite.]

border, across which continuous extension of the surface cannot be directly effected. Only by conceiving each single piece of the border cut loose and displaced along the surface of the ring or cup-glass can one bring it to places of different flexure, at which further continuation of this piece of surface is possible.

In this way the straightest lines of the pseudospherical surface can then also be infinitely prolonged. They do not, like those of the sphere, run back into themselves: rather, as on the plane, there is always only one single shortest line between two given points.[8] But the axiom of parallels does not hold. If a shortest line on the surface is given, with a point outside it, then a whole angular pencil of straightest lines can be passed through the point, all of them failing to intersect the first-named line (even when they are prolonged into the infinite). These are all lines that lie between two straightest lines that bound the pencil. One of these, infinitely prolonged, meets the first-named line in the infinite upon prolongation towards one side; the other, upon prolongation towards the other side.

Such a geometry, which allows the axiom of parallels to fall, was moreover already fully worked out in the year 1829 by *N. I. Lobachevsky*, Professor of Mathematics at Kazan, following the synthetic method of Euclid.*) It turned out that Lobachevsky's system can be carried through consistently and without contradiction just as well as that of *Euclides*. This geometry is in complete agreement with that of the pseudospherical surfaces, as recently developed by *Beltrami*.

We see from this that in the geometry of two dimensions the presupposition that every figure can be moved, without alteration of its dimensions along the surface, characterizes the surface concerned as a plane or sphere or pseudospherical surface. The axiom that between any two points there is always only one shortest line separates the plane and pseudospherical surface from the sphere; and the axiom of parallels separates the plane from the pseudosphere. These three axioms are thus in point of fact necessary and sufficient to characterize the surface to which Euclidean planimetry refers as a plane, in contrast to all other spatial structures of two dimensions.[9]

The difference between the geometry in the plane and that on the surface of a sphere has long been clear and intuitive; but the sense of the axiom of parallels could only be understood after the concept of surfaces that can be bent without stretching had been developed by *Gauß*--and, with its help, the concept of the possible infinite continuation of the pseudospherical surfaces. We, as inhabitants of a space of three dimensions and endowed with instruments of sensation to perceive all these dimensions, can to be sure imagine intuitively to ourselves the various cases in which two-dimensional beings would have to develop their spatial intuition--because to this end we have only to restrict our own intuitions to a

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[8][This statement needs to be qualified--or interpreted: On the ring-surface, one easily sees that there are three infinite classes of geodesics: those that run off both edges (the arcs *ab* are geodesics of this class); those that run off the edge *aa* alone, in two directions; and those that run off the edge *bb* alone, in two directions. Between the geodesics of the latter two classes, there lies a single exceptional case: the mid-circle of the ring, a geodesic that does not run off the realized piece of the surface at all, and that does in fact close back upon itself. Further, among the geodesics that run off only one edge, there are obviously some that wind many times around the ring; and these always intersect geodesics of the first class in many points, and likewise have many self-intersections (as well as multiple meetings with other geodesics of the same type as themselves). As for the champagne glass (or trumpet?), it has no closed geodesics; but the multiple meeting phenomenon occurs on it just as on the ring (so that it is untrue that there is a unique geodesic through two given points). In order to make H.'s statement unqualfiedly true, one has to conceive these surfaces as "multiply wrapped"--like the cylinder or cone. This is perhaps not so easy to visualize: the trouble is that, unlike cylinder or cone, they cannot--within the ambient three-dimensional Euclidean space--be *un*wrapped!]

*) Prinzipien der Geometrie. Kazan, 1829 to 1830.

[9][There is one small correction to be made to H.'s account here: He was unaware--the fact was first pointed out by Felix Klein in 1871, more clearly in 1873--that the sphere is not the only closed surface with a geometry of constant positive curvature: that on the contrary the *projective plane* is a closed surface that admits such a geometry, and one that satisfies the principle that there is a unique geodesic line through any two points. So it is not the case that this principle distinguishes the plane and pseudosphere: on the contrary, the principle is compatible with geometries of all three types (positive, zero, or negative Gaussian curvature).]

narrower domain. To think away intuitions that one possesses is easy; but to imagine sensibly to oneself intuitions of which one has never possessed an analogue is very hard. When, therefore, we pass to space of three dimensions, we are hampered in our capacity of imagination by the construction of our organs and the experiences obtained through them, which conform only to the space in which we live.

But, now, we have another mode for the scientific treatment of geometry. Namely, all the spatial relations we know are measurable: that is, they can be reduced to the determination of magnitudes (lengths of lines, angles, surfaces, volumes). For just this reason, the problems of geometry can also be solved by seeking the methods of calculation that enable one to derive the unknown spatial magnitudes from the known ones. This is done in *analytic geometry*, in which all the structures of space are just treated as magnitudes and determined by other magnitudes. The straight line is defined as the *shortest* line between two points, and this is a magnitude determination. The axiom of parallels says that if two straight lines in the same plane do not intersect (are parallel), the alternate angles, or the opposite angles, with a third line cutting them, are equal in pairs. Or else one substitutes the proposition that the sum of the angles in any triangle is equal to two right angles. These, too, are magnitude determinations.

One can thus also start from this aspect of the concept of space, according to which the position of each point, in relation to some spatial structure (coordinate system) regarded as fixed, can be determined through measurements of some collection of quantities; and one can then examine what particular determinations belong to our space, as it presents itself in the measurements actually carried out--and whether any among these distinguish our space from other magnitudes extended with similar manifoldness.[10] This way was first taken by *B. Riemann* in Göttingen,*) lamentably too soon lost to science. This way has the characteristic merit, that all operations occurring in it are pure calculating determinations of magnitude--so that the danger that accustomed facts of intuition could intrude themselves as necessities of thought falls away entirely.

The number of measurements necessary to give the position of a point is equal to the number of dimensions of the space concerned. In a line, the distance from a fixed point suffices[11]--thus one magnitude; in a surface one must give the distances from two fixed points--in space, from three--in order to fix the position of the point; or we use, as on the earth, geographical longitude, latitude, and altitude above sea-level; or, as ordinarily in analytic geometry, the distances from three coordinate-planes. *Riemann* calls a system of distinctions in which the particular can be determined by *n* parameters an *n-tuply extended manifold*, or a *manifold of n dimensions*. Thus the space we know, in which we live, is a triply extended manifold of points; a surface is a doubly extended, a line a simply extended, and time as well a simply extended manifold. The system of colors also forms a triple manifold, in so far as each color--according to *Th. Young*'s and *Cl. Maxwell*'s investigations[12]--can be represented as the mixture of three basic colors, of each of which a determinate quantum is to be used. With the color top one can actually effect such mixings and measurements.

In the same way, we could regard the domain of the simple tones as a manifold of two dimensions, if we take them as differing only in pitch and volume (and leave aside differences in tone-color). This generalization of the concept is very well suited to throw into relief the way in which space is distinguished from other manifolds of three dimensions. In space, as you all know from everyday experience, we can compare the distance of two points lying one above the other with the horizontal distance of two points in the ground-plane, because we can apply a measuring-rod now to the one, now

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[10][This somewhat obscure phrase is borrowed from Riemann (see text immediately following). By "similar manifoldness"--or "similar multiplicity"--is meant here "the same number of dimensions"; thus our space is, in this terminology, "extended threefold", or "triply extended".

*) "Über die Hypothesen, welche der Geometrie zu Grunde liegen", Habilitationsschrift [*sic;* but to speak precisely, this was Riemann's Habilitationsvortrag--his habilitation *lecture*--not his Habilitationsschrift; the latter dealt with the theory of trigonometric series] of 10. June 1854. Published in Bd. XIII of the Abhandlungen der Königlichen Gesellschaft zu Göttingen.

[11][That is, the distance given with a *sign* specifying the direction ("sense") of the displacement from the fixed point; and correspondingly for the cases that follow.]

[12][Credit H. here with a bit of modesty: the theory of color vision concerned is generally known as the Young-Helmholtz, or the Young-Maxwell-Helmholtz, theory.]

to the other pair. But we cannot compare the distance of two tones of equal pitch and different intensity with that of two tones of equal intensity and different pitch. *Riemann* showed, through considerations of this sort, that the essential foundation of every geometry is the expression by which the distance of two points in arbitrary direction from one another is given--and indeed, in the first instance, the distance of two points infinitely little distant from one another. For this expression he took from analytic geometry the most general form*) that is obtained when the nature of the measurements giving the place of each point is left quite arbitrary. He then showed that the kind of free mobility with unvarying form that belongs to bodies in our space can only subsist if certain magnitudes obtained by calculation**)--which magnitudes, referred to the relations on surfaces, reduce to the *Gaußian* measure of surface curvature--have everywhere the same value. For just this reason, *Riemann* calls these calculated magnitudes, if at a particular place they have the same value for all directions, the measure of curvature of the space concerned at this place. To prevent misunderstandings***), I want to emphasize here that this so-called measure of curvature of space is a magnitude of calculation, found in a purely analytic way, and that its introduction in no way rests upon an intrusion of relationships that would have sense only in sensible intuition. The name is taken, only as an abbreviation of a complicated relationship, from the one case in which the designated magnitude does have corresponding to it a sensible intuition.

what is 'relation that would have only in sens. intuition'?

If, now, this measure of curvature of the space has everywhere the value zero, such a space corresponds everywhere to the axioms of *Euclides*. We can all it in this case a *flat* space, in contrast to other analytically constructible spaces, which one could call curved (because their measure of curvature has a value different from zero). Nonetheless, analytic geometry can be carried out for spaces of the latter sort just as completely and self-consistently as the ordinary geometry of our actually existing flat space.

If the measure of curvature is positive, we obtain the *spherical* space, in which the straightest lines return upon themselves and in which there are no parallels. Such a space would, like the surface of a sphere, be unbounded but not infinitely great. A negative curvature, on the other hand, gives the *pseudospherical* space, in which the straightest lines run out into the infinite, and in every flattest surface, through each point, there can be passed a pencil of straightest lines that do not intersect another given straightest line of that surface.

These last relationships Mr. *Beltrami*****) has made accessible to intuition, by showing how one can map the points, lines, and surfaces, of a pseudospherical space of three dimensions, into the interior of a sphere of the *Euclid*ean space, in such a way that every straightest line of the pseudospherical space is represented in the sphere by a straight line, every flattest surface of the former by a plane in the latter.

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*) Namely, for the square of the distance of two infinitely close points, a homogeneous function of the second degree in the differentials of their coordinates. [For full accuracy, amend: "a homogeneous *positive-definite* function ... ."]

**) They are given by an algebraic expression, composed from the coefficients of the single terms in the expression for the square of the distance of two neighboring points, and from the differential quotients of those coefficients.

***) As, e.g., such a misunderstanding occurs in the above-cited book of Mr. *W. Tobias*. p. 70, inter alia. [It is perhaps worth mentioning that these misunderstandings continued for decades. In Hilbert's circle in Göttingen about the turn of the century, Otto Blumenthal--Hilbert's first doctoral student, famous in that circle for his comic verse--put the case in the following couplet:

Die Menschen fassen kaum es,
Das Krümmungsmaß des Raumes.

A lame English version--the doggerel seems hardly imitable--might be:

Men's minds can scarce embrace
The curvature of space.]

****) Teoria fondamentale degli Spazii di Curvatura costante. Annali di Matematica. Ser. II, Tom. II, Fasc. III, pp. 232-255.

The surface of the sphere itself then corresponds to the infinitely distant points of the pseudospherical space; the different parts of that space are, in the image in the sphere, the more diminished the nearer they lie to the surface of the sphere--and more strongly diminished in the radial direction than in directions perpendicular thereto. Straight lines in the sphere which intersect only outside its surface correspond to straightest lines of the pseudospherical space that never intersect.

Thus it turns out that space, regarded as a domain of measurable magnitudes, by no means corresponds to the most general concept of a manifold of three dimensions, but rather contains more special determinations, which are stipulated by the condition of perfectly free mobility of solid bodies, with unvarying form, towards all places and under all possible changes of orientation, and further by the special value of the measure of curvature--which, for the actually existing space, is to be set equal to zero, or at least does not differ appreciably from zero. This last stipulation is given in the axioms of straight lines and of parallels.

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While *Riemann* entered upon this new domain from the most general basic questions of analytic geometry, I myself arrive, partly through investigations concerning the spatial representation of the system of colors (thus through comparison of one triply extended manifold with another), partly through investigations concerning the origin of our visual estimation of measurements in the visual field, at considerations similar to *Riemann's*. While the latter proceeds from the algebraic expression mentioned above, which represents the distance of two infinitely close points in most general form, as his basic assumption, and derives from it the propositions about the mobility of rigid spatial structures, I on the other hand have proceeded from the fact of observation that in our space the motion of rigid spatial structures is possible with the degree of freedom that we know; and I have derived from this fact the necessity of that algebraic expression which *Riemann* posits as an axiom. The assumptions that I had to place at the basis of the calculation were the following.

First, to make calculating treatment possible at all, it must be presupposed that the position of any point A with respect to certain spatial structures regarded as invariable and fixed can be determined by measurements of some collection of spatial magnitudes--whether they be lines, or angles between lines, or angles between planes, etc. As is well known, one calls the parameters needed to determine the position of the point A its *coordinates*. The number of coordinates needed in general for the complete specification of the position of any point determines the number of dimensions of the space concerned. It is further presupposed that in the motion of the point A the spatial magnitudes used as coordinates vary continuously.

Second, the definition of a rigid body--or of a rigid system of points--is to be given, as is necessary in order to be able to undertake comparison of spatial magnitudes through congruence. Since we may not here presuppose any special methods of measuring spatial magnitudes, the definition of a rigid body can only be given through the following distinctive mark: Between the coordinates of every pair of points belonging to a rigid body there must hold an equation expressing a spatial relation between the two points that is unchanged in every motion of the body (a relation that finally turns out to be their distance); and this equation must be the same for all congruent point-pairs. But as to the meaning here of "congruence", we call two point-pairs congruent if they can be brought successively into coincidence with the same spatially fixed point-pairs.[13]

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[13][H.'s exposition of this postulate is rather cryptic, and a fuller explanation seems desirable: We have to conceive, on the one hand, a "space" S, having the structure of an n-dimensional differentiable manifold (in the case of "physical space", n of course is 3). The points of S, then, are "fixed points of space". On the other hand, we conceive a (rigid) "body" B as itself a set of points ("bodily points"). A *spatial position of the body* B means an assignment to each bodily point P of B of a point in S: the spatial point "occupied" by P in that position of B. H. in effect postulates the following: (1) To each pair of distinct points P, Q, of B, there corresponds a function, F_{PQ} of pairs of points of S, such that, for any given points p, q, of S, there is a position of B in which P occupies p and Q occupies q if and only if $F_{PQ}(p,q) = 0$. If we think of p and q as given by their coordinates in a suitable coordinate system, this is an equation among the $2n$ coordinates of these two points; it is the "equation between the coordinates of every pair of points that is unchanged in every motion of the body" to which H. refers. (2) If P, Q, P', Q', are points of B such that for *some* pair of points p, q, of S we have $F_{PQ}(p,q) = 0 = F_{P'Q'}(p,q)$--i.e., if (P,Q) and (P',Q') can,

Despite its seemingly so indefinite formulation, this stipulation has very strong consequences, because upon increasing the number of points we increase the number of equations much faster than the number of coordinates of points governed by those equations. Five points--*A, B, C, D, E*--give ten different point-pairs:

$$\begin{array}{r} AB, AC, AD, AE, \\ BC, BD, BE, \\ CD, CE, \\ DE, \end{array}$$

therefore ten equations,[14] which in three-dimensional space contain fifteen variable coordinates--six of which, however, must remain freely disposable if the system of five points is to be freely mobile and rotatable. Thus only nine coordinates can be determined by those ten equations as functions of those six variable ones. With six points we get fifteen equations for twelve variable magnitudes, with seven points twenty-one equations for fifteen magnitudes, etc. But, now, from n independent equations we can determine n magnitudes occurring in them. If we have more than n equations, the supernumerary ones must themselves be derivable from the first n of them. From this it follows that those equations that hold between the coordinates of every point-pair of a rigid body must be of a special kind, so that, when (in three-dimensional space) they are satisfied for nine of the point-pairs formed from any five points, the equation tor the tenth pair follows identically from them. On this circumstance it rests, that the cited assumption as definition of rigidity suffices to determine the type of the equations that hold between the coordinates of two points connected rigidly with one another.

Third, it has proved that yet another special property of the motion of rigid bodies must be set as a fact at the basis of the calculation--a property so familiar to us, that without this investigation it might perhaps never have occurred to us to regard it as something that might not be so. If, namely, in our space of three dimensions, we hold two points of a rigid body fixed, the body can then still just execute rotations about their straight connecting-line as axis. If we turn the body once completely around, it comes precisely into the position it occupied initially. Now, that rotation without reversal of direction carries every rigid body recurrently back into its initial position has to be specifically stipulated. A geometry would be possible in which this was not so. This is easiest to see for the geometry of the plane.[15] Imagine that on every rotation of every plane figure its linear dimensions were to grow proportionally to the angle of rotation; then after a full rotation of 360° the figure would no longer be congruent to its initial state. However, every other figure that was congruent to it in the initial position would be able to be

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successively, be brought to occupy one particular "spatially fixed point-pair"--then the functions F_{PQ} and $F_{P'Q'}$ are identical ("congruent point-pairs have the same equation"). In addition, H. assumes (3) the existence of a point P of B such that in any spatial position of B all the points in some spatial neighborhood of the point occupied by P are themselves occupied by bodily points (P is then an *interior point* of B, and always occupies an interior point of any spatial position of B); and he assumes (4) the postulate of free mobility--which we shall not attempt to formulate here with full precision (in point of fact, H. himself failed in his attempt to do this--a defect that was pointed out, and repaired, by Sophus Lie many years later [preliminary report, 1886; with full details, 1890]), but which roughly says: (a) that P can be brought to occupy any spatial point; (b) that any "line" L through P in B can then be made "tangent" to any spatial line through that spatial point; (c) that any "surface" F through L in B can then be made "tangent" to any spatial surface through that spatial line; and so on, through structures of dimension n - 1 (i.e., less by one than the dimension-number of the manifold S itself); and that this series of specifications determines the position of B uniquely. (To be more accurate, for the determination to be unique the "elements"--line, surface, etc.--named in this series of specifications must be *oriented* elements: a line *with a given "sense" or direction along the line*, a surface *with a given sense of rotation*, etc.) ?

14[This presupposes--in the notation of the preceding note--that we always have $F_{PQ} = F_{P'Q'}$.]

15[It is, in fact, true *only* for the geometry of the plane: as Klein conjectured, and Lie proved, the additional assumption made by H. can be proved from his other assumptions when the dimension n is greater than 2.] spinors?

made congruent to it in the second position as well, if the other figure also is turned through 360°. A consistent system of geometry would be possible under this assumption too--a system that does not fall under the *Riemann*ian form.

On the other hand, I have shown[16] that the enumerated three assumptions are together sufficient to ground the starting-point *assumed* by *Riemann* for his investigation--and, with it, all further results of his work that bear upon the distinction of the various spaces according to their measure of curvature.

It could now be asked, further, whether the laws of motion and of its dependence on the moving forces can also be transferred without contradiction to the spherical and pseudospherical spaces. This investigation has been carried out by Professor *Lipschitz**) in Bonn. It is indeed possible to carry over the comprehensive formulation of all the laws of dynamics, the *Hamilton*ian principle, directly to spaces whose measure of curvature is non-zero. Thus on this side as well the deviant systems of geometry fall into no contradiction.

But now we shall have to ask where these special prescriptions come from, that characterize our space as a flat space; since they have proved not to be contained in the general concept of an extended magnitude of three dimensions, with free mobility of the bounded structures within it. *Necessities of thought*--flowing from the concept of such a manifold and of its measurability, or from the most general concept of a rigid structure and of its free mobility within the manifold--they are not.

We now want to examine the antithetical assumption about the origin of these prescriptions: namely, to investigate the question whether they are of *empirical origin:* whether they are to be derived from facts of experience--to be demonstrated, or to be tested and perhaps even refuted, by such facts. The last eventuality would imply the following: that we should have to be able to imagine series of observable facts of experience, through which another value of the measure of curvature would be indicated than that which the flat space of *Euclides* possesses. But if spaces of other sort are in this sense imaginable, this very fact would refute the view that the axioms of geometry are necessary consequences of an a priori given transcendental form of our intuition in the *Kant*ian sense.

The difference of the *Euclid*ean, spherical, and pseudospherical geometry rests (as remarked above) on the value of a certain constant, which *Riemann* calls the measure of curvature of the space concerned, and whose value must be equal to zero if the axioms of *Euclides* hold. If it is not equal to zero, then triangles of large surface-content would have to have a different angle-sum from that of small ones--a larger sum in spherical space, a smaller sum in pseudospherical space. Further, geometrical similarity of large and small bodies or figures is possible only in *Euclid*ean space. All systems of practically executed geometrical measurements by which the three angles of large rectilinear triangles are separately measured, and especially all systems of astronomical measurements which yield for the parallaxes of the immeasurably distant fixed stars the value zero (in pseudospherical space the infinitely distant points would have to have still positive parallaxes), empirically confirm the axiom of parallels, and show that in our space and by the use of our methods of measurement the measure of curvature of space appears as indistinguishable from zero. Of course one must, with *Riemann,* raise the question whether this would not perhaps be otherwise if instead of our limited base-lines--the biggest of which is the major axis of the earth's orbit--we could use bigger ones.

But we must not forget, in this, that all geometrical measurements finally rest upon the principle of congruence. We measure distances of points by moving compasses or measuring-rod or measuring-chain

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[16]["Über die Thatsachen, die der Geometrie zum Grunde liegen". Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, 1868. Reprinted in Wissenschaftliche Abhandlungen von Hermann Helmholtz, Bd. II, pp. 618-639.--H.'s analysis leaves some logical gaps; his point of view was taken up later, and his theory perfected (cf. n. 13 above), in a very famous work by Sophus Lie: "Über die Grundlagen der Geometrie". Verhandlungen der Sächsischen Gesellschaft der Wissenschaften, Bd. 42 (1890), pp. 284-321, 355-418; also S. Lie and F. Engel, Theorie der Transformationsgruppen, Bd. 3 (1893), pp. 393-543. (This work formed a principal part of a group of investigations that won for Lie the first award of the Lobachevsky Prize.)]

*) Untersuchungen über die ganzen homogenen Functionen von *n* Differentialen. *Borchardt*'s Journal für Mathematik, Bd. LXX, p. 71 and Bd. LXII, p. 1.--Untersuchung eines Problem der Variationsrechnung, ibid., Bd. LXXIV.

to them. We measure angles by bringing divided circle or theodolite to the vertex of the angle. Besides this, we determine straight lines through the paths of light-rays, which according to our experience are rectilinear; but that light is propagated along shortest lines (so long as it remains within a single refractive medium) is a proposition that would also admit of transfer to spaces of different measure of curvature. Thus all our geometrical measurements rest upon the presupposition that the measuring instruments we treat as rigid really are bodies of invariable form--or at least that they suffer no other sorts of change of form than those that we know, as e.g. the small deformations resulting from change of temperature, or from the difference in the action of gravity in different places.

When we measure, we only effect with the best and most reliable aids known to us the same thing that we ordinarily ascertain through observation by visual estimation, judgment by touch, or pacing off. In these cases, our own body with its organs is the measuring instrument that we carry about in space. Our compasses are now the hand, now the limbs; or the eye, turning towards all directions, is our theodolite, with which we measure arc-lengths or surface-angles an the visual field.

Every comparison of magnitudes of spatial relations, whether by estimation or by measurement, thus proceeds from a presupposition about the physical behavior of certain natural bodies, whether our own body or the measuring instruments employed; a presupposition which, moreover, may possess the highest degree of probability, and may stand in the best agreement with all other physical relationships known to us, but which in any case reaches beyond the domain of pure spatial intuition.

Indeed, one can describe a specific behavior of the bodies that appear to us as rigid, upon which behavior the measurements in Euclidean space would turn out as if they had been performed in pseudospherical or spherical space. In order to see this, I first remind you that if all the linear dimensions of the bodies surrounding us, together with those of our own body, were changed in the same proportion (e.g. all diminished to half, or all increased to double), we should be entirely unable to notice such a change through our means of spatial intuition. But the same would also be the case if the expansion or contraction were different in different spatial directions--provided that our own body changed in the same way, and provided further that a turning body assumed, at each instant, without suffering or exerting mechanical resistance, that degree of expansion of its various dimensions which corresponds to its instantaneous position. Think of the image of the world in a convex mirror. The familiar silvered globes often set up in gardens show the essential phenomena of such an image, though distorted by some optical anomalies. A well made convex mirror of not too large aperture shows the mirror-image of each object before it as bodily in appearance and with a determinate orientation and at a determinate distance behind the mirror's surface. But the image of the far horizon and of the sun in the sky fall at a limited distance behind the mirror--a distance equal to the mirror's focal length. Between these images and the surface of the mirror are contained the images of all the other objects before the mirror--but in such a way that the images are the more diminished and the more flattened, the farther their objects lie from the mirror. The flattening--that is, the diminution of the dimension of depth--is relatively more considerable than the diminution of the surface-dimensions. Nevertheless, every straight line of the outside world is represented by a straight line in the image, every plane by a plane. The image of a man who measures off with a measuring rod a straight line receding from the mirror would shrink ever more, the more the man himself recedes; but the man in the image, with his likewise shrinking measuring-rod, would count exactly the same number of centimeters as the man in the real world; in general, all geometrical measurements of lines or angles carried out with the regularly varying mirror-images of the real instruments would yield exactly the same results as in the outside world; all congruent figures would conform just as well, in the images, upon actual superposition of the bodies concerned, as they would in the outside world; all lines of sight of the outside world would be replaced by straight lines of sight in the mirror. In short, I do not see how the men in the mirror should make out that their bodies are not rigid bodies and their experiences not good instances of the correctness of the axioms of *Euclides*. But if they could look out upon our world as we look in upon theirs, without being able to step over the boundary, they would have to declare our world to be the image made by a convex mirror, and to speak of us just as we speak of them; and if men of both worlds could converse with one another, then so far as I see neither would be able to convince the other that his are the true relationships and the other's the distorted ones; indeed, I cannot admit that such a question would have a sense at all, so long as we do not introduce any mechanical considerations.

because all the bodies will be as rigid as ours, and will have same relations (metrical or otherwise) to each other

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another Euc. sp. since the reflected bodies will not be rigid relative to ours

Now Mr. *Beltrami*'s mapping of pseudospherical space into a solid sphere of *Euclid*ean space is of entirely similar sort; except that the background surface is not, as in the convex mirror, a plane, but the surface of a sphere, and the proportion in which images contract as they approach that surface has a different mathematical expression. Thus if one supposes that within the sphere--for whose interior the axioms of *Euclides* hold--bodies move in such a way that as they recede from the center they always contract (analogously to the images in the convex mirror), and contract in such fashion that their images (by Beltrami's) construction) in pseudospherical space retain unaltered dimensions, then observers whose own bodies were also regularly subject to this alteration would obtain from the geometrical measurements they could effect just the same results as if they themselves lived in pseudospherical space.

From this we can even proceed a step further: we can deduce how the objects of a pseudospherical world would appear to an observer whose visual estimation and spatial experiences have been developed like ours in flat space, if such an observer could enter such a world. The observer would continue to regard the lines of light-rays or the lines of sight of his eye as straight lines, just as they are in flat space and as they are in fact in the spherical image of pseudospherical space. The visual image of the objects in pseudospherical space would therefore make the same impression upon him as if he sere situated at the center of the *Beltrami* spherical model. He would believe himself to see the most distant objects of this space ranged about him at a finite distance*)--let us say for example a hundred feet away. But if he went towards these distant objects, they would expand before him--and more so in depth than in surface; whereas behind him they would contract. He would perceive that his judgment by visual estimation had been false. If he saw two straight lines which by his estimate were parallel up to this distance of 100 feet (where the world seems to him to end), then, going along those lines, he would perceive that, upon this expansion of the objects he approaches, the lines draw apart from one another, and the more so the farther he advances along them; behind him, on the contrary, their distance would seem to dwindle, so that as he goes forward they would seem to him ever more divergent and ever farther from one another. But two straight lines that seem, from the initial standpoint, to converge towards a single point of the background one hundred feet away, would always seem so, no matter how far he went; and he would never reach their point of intersection.

Now, we can obtain quite similar images of our actual world, if we take before our eyes a large [concave] [17] lens of suitable negative focal length--or, alternatively, two [concave] spectacle-glasses (which must be cut somewhat prismatically, as if they were pieces of a connected larger lens). Such glasses show us distant objects brought closer (just as does the above-mentioned convex mirror)--the farthest objects being brought up to the distance of the focus of the lens. If we move about with such a lens before our eyes, expansions occur of the objects we approach, quite similar to those I have described for the pseudospherical space.[18] If now someone takes such a lens before his eyes--and not of hundred foot focal length, but a much stronger one, of focal length only sixty inches--he may notice at first that he sees objects as if brought nearer. But after a little moving back and forth the illusion vanishes, and he judges distances correctly despite the false images. We have every reason to suppose that it would very soon fare the same with us in pseudospherical space as it does in fact, within a few hours, with a new wearer of spectacles: in short, the pseudospherical space would seem to us comparatively not very

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*) The reciprocal negative square of this distance would be the measure of curvature of the pseudospherical space.

[17][In H.'s text one finds "convex"; but a lens of negative focal length--one that produces the effects he describes--is concave. The slip is puzzling; perhaps it was then standard usage to refer to a *concave* lens as a "*convex* one of negative focal length" (as one might say "minus five feet to the right", and mean five feet to the left.]

[18][It should be noted that H. is speaking here of a *qualitative* analogy. The *quantitative* relationships for what would be seen through such concave eyeglasses are not those of the Beltrami model, and do not lead to pseudospherical geometry: exactly as with the convex mirror--and for the very same reason--the geometry that results remains Euclidean. (A genuine--although "ideal" rather than physically constructible--simulation of pseudospherical space is described by H. Poincaré, Science and Hypothesis, ch. iv.]

strange at all--only in the beginning should we find ourselves subject to illusions in the estimate of size and distance of far-away objects by means of our visual impression of them.

The contrary illusions would result in a spherical space of three dimensions, should we enter it with the visual judgment developed in Euclidean space. We should regard the more distant objects as farther away and larger than they are; we should find, on going towards them, that we reach them quicker than we expected from their visual image. But we should also see objects in front of us that we can fixate only by diverging our lines of sight: this would be the case for all objects farther from us than a quadrant of a great circle. This sort of aspect would hardly seem very extraordinary to us; for we could produce the same effect for terrestrial objects too, by taking before one eye a weakly prismatic glass whose thicker side is turned towards the nose. In this case, too, we have to make our eyes diverge in order to fixate distant objects, This arouses a certain feeling of unwonted strain in the eyes, but does not noticeably alter the aspect of the objects viewed in this way. The most singular part of the aspect of the spherical world, however, would be constituted by the back of our own head: all our lines of sight, so far as they can pass freely between other objects, would come together again there; and so the back of our head would entirely fill the outermost background of our whole perspectival picture.

A further remark by way of qualification is required, to be sure: just as a small flat elastic disk--for instance, a small flat slab of rubber--can be fitted to the mildly arched surface of a sphere only with some relative contraction of its periphery and expansion of its central part, so our body, having grown in Euclidean flat space, could not pass over into a curved space without undergoing similar expansions and compressions of its parts; and the connection of the parts could be preserved therefore only in so far as their elasticity allowed a yielding without tearing or breaking. The type of expansion would have to be just the same as if we conceived a small body to be at the center of *Beltrami*'s sphere, and we then passed from this to its pseudospherical or spherical image.[19] In order for such a passage to appear as possible, it must always be supposed that the body is sufficiently elastic, and small enough in comparison with the real or imaginary radius of curvature of the curved space into which it is to pass.

This will be enough to show how, upon the way adopted, one can derive from the known laws of our sensible perceptions the series of sensible impressions that a spherical or pseudospherical world would give us, if such a world existed. Here again we nowhere come upon an illogicality or impossibility--just as little as in the calculating treatment of the measure-relationships. We can depict for ourselves the appearance, in all directions, of a pseudospherical world, just as well as we can develop the concept of such a world. We therefore cannot admit that the axioms of geometry are founded in the given form of our capacity of intuition, or are in any way connected with such a form.

It is different with the three dimensions of space. Since all our means of sensible intuition extend only to a world of three dimensions, and the fourth dimension would be not merely an alteration of what is present but something completely new, we therefore find ourselves simply by virtue of our bodily organization in a condition of the absolute impossibility of imagining to ourselves a mode of intuition of a fourth dimension.

I should like now, in conclusion, to stress yet again that the geometrical axioms are not propositions that belong only to the pure theory of space. They speak, as I have already remarked, of magnitudes. One can only talk of magnitudes when one knows and has in mind some procedure by which these magnitudes can be compared, subdivided, and measured. All spatial measurements, and therefore in general all concepts of magnitude applied to space, thus presuppose the possibility of the motion of

[19][The expression is a little careless: Beltrami's construction does not give us an image of spherical space. As to the "passage to the pseudospherical image", what is meant more fully is this: We begin with a body at the center of Beltrami's sphere, and we think of the distances among the parts of that body as those given by the *Euclidean* geometry of the interior of the sphere; then we replace those Euclidean distances by the pseudospherical ones--those that hold in the pseudospherical space of which the interior of the sphere is a model; or, in other words, those that would be measured within the sphere by the appropriately expanding and contracting measuring-rods. On this change of the distances between its parts, the outer parts of the body will expand, and proportionally more in the radial than the transverse directions. This is to be thought of, not as a mere redefinition of "distance", but as an *actual* change, that affects the *elastic stresses* within the body (and, if too drastic, could result, as H. has said, in breakage).]

spatial structures whose shape and size can be regarded as invariable despite their motion. Such spatial forms one does, to be sure, usually designate in geometry as merely geometrical bodies, surfaces, angles, lines--because one abstracts from all other differences of physical and chemical sort that natural bodies show; but one still preserves one physical property of these forms: their rigidity. For the rigidity of bodies and spatial structures, however, we have no other criterion than that they manifest upon mutual superposition--at all times and all places and after any rotations--always the same congruences. But whether the superposed bodies have not both altered in the same way is something which--in a purely geometrical way--we cannot at all decide.

If we found it useful to some end or other, we could in an altogether cogent way consider the space in which we live as the apparent space behind a convex mirror, with foreshortened and contracted background; or we could consider a bounded sphere of our space, beyond whose boundaries we perceive nothing more, as the infinite pseudospherical space. We should then only have to ascribe to the bodies that appear to us as rigid--and likewise at the same time to our own body--the corresponding expansions and diminutions. (We should, to be sure, also have to revise completely the system of our mechanical principles: for already the proposition that any moving point upon which no forces acts continues to move in a straight line with unchanged velocity fails to hold for the image of the world in a convex mirror. The path, indeed, would still be straight; but the velocity would vary with the place.)[20]

Thus the geometrical axioms do not speak at all about the relationships of space alone; rather they speak at the same time about the mechanical behavior of our most rigid bodies in their motions. One could (it must be granted) take the concept of a rigid geometric spatial structure itself as a transcendental concept, which is formed independently of experience and to which experience does not have to correspond, [21] as our natural bodies actually do not correspond altogether perfectly and without variation even to those concepts that we have abstracted from them by the way of induction. With the addition of such a concept of rigidity, conceived merely as ideal, as strict *Kant*ian could then regard the geometrical axioms as propositions given a priori through transcendental intuition--propositions that could neither be confirmed nor refuted by any experience, because only with their help would one be in a position to decide whether some particular bodies are to be regarded as rigid. But then we should have to assert that under this conception the geometrical axioms would not be synthetic propositions in *Kant*'s sense at all. For they would then only say something that follows analytically from the concept of the rigid geometrical structure necessary for measurement--since only such structures as satisfy those axioms could be admitted to be rigid.

But if we add to the geometrical axioms propositions that relate to the mechanical properties of natural bodies--if only just the proposition of inertia, or the proposition that the mechanical and physical properties of bodies under otherwise constant influences cannot depend upon the place in which they happen to be--then such a system of propositions receives a real content, which can be confirmed or refuted by experience; but for just this reason they can then be learned from experience.

For the rest, it is of course not my intention to assert that only through meticulously executed systems of precise geometrical measurements has mankind achieved intuitions of space that correspond to the axioms of *Euclides*. It must rather have been that a series of everyday experiences--in particular, the intuition of the geometrical similarity of larger and smaller bodies, which is possible only in flat space--led to the rejection of any geometrical intuition that contradicted this fact. For this, recognition of the conceptual connection between the observed facts of geometrical similarity and the axioms was not needed, but only an intuitive knowledge of the typical conditions of spatial relationships, achieved

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[20][This last statement requires some care in its interpretation, if H. is not to be taken to assert something false. The reader should consider (reflect upon?) a game of baseball, or billiards, and its image in the convex mirror: which of these--and how considered--would require a revision of mechanical principles?]

[21][Note that H. here uses the word "transcendental" in a way that diverges seriously from the sense tin which that word is used by Kant: for the latter, a transcendental concept is one to which, a priori, any possible experience is subject; a transcendental principle is one that holds for all experience, either because it results from the very form of our sensible intuition, or because it is a conceptual condition that the sensuous flow must conform to if it is to rank as objective experience.]

through numerous and exact observations of such relationships: such an intuition as the artist has of the objects to be portrayed, by means of which he decides with sureness and finesses whether a ventured combination corresponds to the nature of the object to be portrayed, or does not. This, to be sure, we have no other name for in our language than "intuition"; but the knowledge is an empirical knowledge, gained through the accumulation and reinforcement of homogeneous recurrent impressions in our memory--not a transcendental form of intuition given prior to all experience. That such empirically won intuitions of a typical lawful behavior or state of affairs, not yet worked through to the clarity of a distinctly expressed concept, have often enough imposed upon the metaphysicians as propositions given a priori, is something there is no need to argue further here.