Kant, Riemann and Reichenbach on Space and Geometry

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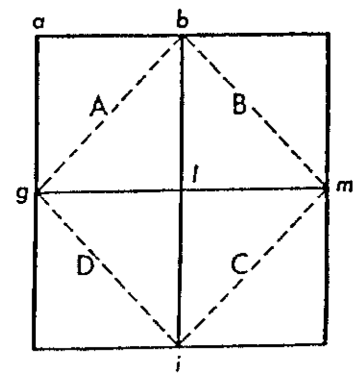
Classic examples of ostensive geometrical constructions are used to clarify Kant's account of how they provide knowledge of claims about rigid bodies we can observe and manipulate. It is argued that on Kant's account claims warranted by ostensive constructions must be limited to scales and tolerances corresponding to our perceptual competencies. This limitation opens the way to view Riemann's work as contributing valuable conceptual resources for extending geometrical knowledge beyond the bounds of observation. It is argued that neither Reichenbach's descriptions of non-Euclidean visualization nor his arguments for conventionalism about geometry undercut this view of Kant's account of geometrical knowledge.

I. Kant on Ostensive Constructions and Space

1. Plato's slave boy passage in the *Meno*

Readers of the *Meno* are confronted with Socrates' elicitation from the slave boy of a constructive proof that a square with twice the area of a given one will have sides of length equal to that of the diagonal of the original. Socrates encourages the slave boy to add three more squares to the original to generate a square of 4 times the area of the original. Then he makes the key suggestion that they consider the diagonals.



The slave boy is made to recognise that the smaller square formed by the diagonals of these 4 has an area equal to half of the total.

> Socrates: Does not each of these lines cut each of the spaces, four spaces, in half? Is that right? (Rouse, 49)

The slave boy is, thereby, brought to understand that the square made from the diagonal has twice the area of the original.

The demonstration is very convincing, so much so that Plato used it to argue that the slave boy (and all the rest of us) must have already known geometry

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before our experiences in this life. The warrant provided, without any explicit appeal to previous learning experiences, seems better than what would count as inductive support from a whole series of successful trials construed as, merely, empirical experiments.

Plato is able to appeal to the incommensurability of the diagonal with the side of a square to make this construction essential, in a way that it would not have been if there had been a rational number to multiply the length of the original side to double the area. According to Torretti (1984, 10), the discovery of incommensurables brought about the end of the Pythagorean program of reducing elements of things to elements of numbers. Eudoxus, who was associated with Plato's academy, invented a theory of proportions that would allow such an irrational quantity to be approximated by constructing a series of ratios of rational numbers. (see Euclid bk. V, Heath Vol.2, 112-186) Plato's use of the diagonal, thus, allowed him to associate his discussion with the very latest and most impressive developments in greek mathematics.

There is an interesting analogy between the absence of numbers that might have made the appeal to ostensive construction unnecessary, had they been available, and the absence of predicate logic, which, Friedman (1985, reprinted in Posy 1992 and Friedman 1992) has argued, made appeal to ostensive constructions in geometry essential in Kant's day. Had irrational numbers closed under root operations been available, algebra could have been employed to give the answer 'the square root of 8' as the length of the side required to double the area of a square with sides of length 2 units. Such a proof might have been more difficult to plausibly elicit from an untutored slave boy. More importantly, whatever sense in which it would make appeal to the ostensive proof unnecessary must be compatible with the fact that getting agreement with the ostensive geometrical constructions is part of what made the introduction of the new numbers and operations count as adequate.

2. Euclid's proof of the angle sum theorem

This is one Kant's own examples. Here is some of his discussion from the chapter of the *Critique of Pure Reason* he calls "The Discipline of Pure Reason" (A 716-717, B 744-745)[1]:

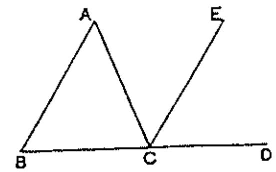
> Suppose a philosopher be given the concept of a triangle and he be left to find out, in his own way, what relation the sum of its angles bears to a right angle. He has nothing but the concept of a figure enclosed by three straight lines, and possessing three angles. However long he meditates on this concept, he will never produce anything new. He can analyze and clarify the concept of a straight line or of an angle or of the number three, but he can never arrive at any property not already contained in these concepts. Now let the geometrician take up these questions. He at once begins by constructing a triangle. Since he knows that the sum of two right angles is exactly equal to the sum of all the adjacent angles, which can be constructed from a single point on a straight line, he prolongs one side of his triangle and obtains two adjacent angles, which together are equal to two right angles. He then divides the external angle by drawing a line parallel to the opposite side of the triangle, and observes that he has thus obtained an external adjacent angle which is equal to an internal angle—and so on. In this fashion, through a chain of inferences guided throughout by intuition, he arrives at a fully evident and universally valid solution of the problem.

Here is proposition 32 from Euclid's elements (Heath, 316):

> In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.

The procedure Kant attributes to the geometrician is that of Euclid's proof.

> Let *ABC* be a triangle, and let one side of it *BC* be produced to *D*;
> I say that the exterior angle *ACD* is equal to the two interior and opposite angles *CAB*, *ABC*, and the three interior angles of the triangle *ABC*, *BCA*, *CAB* are equal to two right angles.
> For let *CE* be drawn through the point *C* parallel to the straight line *AB*. [I.31]
> Then, since *AB* is parallel to *CE*, and *AC* has fallen upon them, the alternate angles *BAC*, *ACE* are equal to one another. [I,29]
> Again, since *AB* is parallel to *CE*,
> and the straight line *BD* has fallen upon them,
> the exterior angle *ECD* is equal to the interior and opposite angle *ABC*. [I.29]
> But the angle *ACE* was proved equal to the angle *BAC*;
> Therefore the whole angle *ACD* is equal to the two interior and opposite angles *BAC*, *ABC*.
> Let the angle *ACB* be added to each;
> Therefore the angles *ACD*, *ACB* are equal to the three angles *ABC*, *BCA*, *CAB*.
> But the angles *ACD*, *ACB* are equal to two right angles; [I.13]
> therefore the angles *ABC*, *BCA*, *CAB* are also equal to two right angles.
> Therefore etc.
> Q. E. D. (Heath, 316-317)



Euclid begins by extending side *BC* to *D* to form straight line *BCD*.[2] Next, line *CE* is drawn through point *C* parallel to the straight line *AB*. This construction is provided for by proposition I.31, to which Euclid explicitly appeals.[3] Euclid then appeals to proposition I.29 [4] to argue that angle *BAC* equals angle *ACE* and that angle *CBA* equals angle *DCE*. Given these equalities, the exterior angle *ACD* equals the sum of the opposite interior angles *CBA* and *BAC*. Adding interior angle *BCA* to the exterior angle *ACD* is, thus, equal to the sum of the three interior angles *BCA*, *CBA*, *BAC*. But the angle *BCD* formed by adding *BCA* to the exterior angle *ACD* is equal to two right angles. (Here Euclid appeals to proposition I. 13).[5] Therefore, the sum of the three interior angles *BCA*, *CBA*, *BAC* is also equal to two right angles.

3. Philip Kitcher's objection.

Philip Kitcher (1975, reprinted in Posy 1992) suggests that the appeal to the constructed diagram, in Kant's discussion of Euclid's proof, introduces a problem of distinguishing those features that are to count as properties of all triangles from accidental features of the particular constructed triangle. His specific example is a proof of the side sum property.

> Suppose that I construct a scalene triangle. From my figure I can generalize that all triangles have the side-sum property (the property that the sum of the lengths of any two sides is greater than the length of the third); but I must not infer that all triangles are scalene. Why is the one inference legitimate and not the other? (1992, 123)

Kitcher begins his discussion of Kant's answer by considering the following passage, which immediately precedes the paragraph (quoted above) in which Kant introduces Euclid's proof.

> —mathematics can achieve nothing by concepts alone but hastens at once to intuition, in which it considers the concept *in concreto*, though not empirically, but only in an intuition which it presents *a priori* that is which it has constructed,and in which whatever follows from the universal conditions of the construction must be universally valid of the object of the concept thus constructed. (A 715-716, B 743-744)

He points out that this passage is illuminated by Kant's discussion of schemata:

> it is schemata, not images of objects, which underlie our pure sensible concepts. No image could ever be adequate to the concept of a triangle in general. (A 140-141, B 180)

According to Kitcher, the schema is the rule for constructing triangles and Kant's solution to the problem is to generalize only those features of the image on which the rule has pronounced.

Kitcher does not find this answer adequate. He suggests that it would make the appeal to the particular construction unnecessary.

> The trouble with this reply is that it seems to make the exhibition of a particular triangle in intuition unnecessary. For if all we are allowed to do is to draw out features of triangles prescribed by the schema of the concept "triangle," then we can do this by conceptual analysis alone. (124)

He goes on (124) to suggest that the way for Kant to answer this further objection is to consider an analogy to learning the geometry of a surface by exploring properties of figures drawn on it. Those properties not analytically contained in the concept "triangle", but which nevertheless are features of all triangles on the surface he calls *S* properties. (124-125)

Like the answer to the previous problem, he takes this answer to raise problems of its own.

> The upshot of this is that, to recognize something as an *S*-property we already have to know what the properties of space are. Without knowing that we were not confronting the Reichenbachian space we could not take the angle-sum property to be an *S*-property. The intuition is supposed, however, to show us that we are confronting Euclidean space. But we cannot draw this conclusion until we have distinguished the S-properties. Clearly the account is turning in circles. (125-126)

The Reichenbachian space Kitcher refers to is a plane with a protruding hemisphere. (Reichenbach 1958, 11) A triangle on the hemisphere would not have angles that summed to 180 degrees.

4. What can be said to respond to Kitcher?

Let us begin with his initial objection. How are we to avoid generalizing accidental features of the particular triangle we construct? Kitcher interprets Kant's account of schemata as an attempt to provide an answer to the problem of how to avoid generalizing such indifferent features as the particular magnitudes of the angles. I interpret Kant's remark in the schematism passage as an appeal to Euclid's practice to help clarify schemata by pointing out how it is schemata and not mere images that are appealed to in geometrical constructions.

Euclid's proof of the angle sum theorem carefully avoids appeal to any features of the constructed triangle that are not generalizable to the construction of any plane triangle. According to Euclid's postulates the further constructions, extending a side beyond a given angle followed by constructing a straight line from the base of that angle parallel to the opposite side, are ones that can be carried out on any angle and side of any constructed plane triangle. The proof is carefully formulated to appeal only to those features unavoidably generated by any such constructions. For Euclid, this is what makes the angle sum theorem hold for any plane triangle. His proof demonstrates how to produce, for any constructed plane triangle, further constructions that generate the compelling recognition that the constructed exterior angles are equal to the opposite interior angles and together with the remaining angle make up two right angles.

Kant's discussion of pure intuition in section I of "The Discipline of Pure Reason" appeals to just this aspect of Euclid's proofs.

> Thus I construct a triangle by representing the object which corresponds to this concept either by imagination alone, in pure intuition, or in accordance therewith also on paper, in empirical intuition—in both cases completely *a priori*, without having borrowed the pattern from any experience. The single figure which we draw is empirical, and yet it serves to express the concept, without impairing its universality. For in this empirical intuition we consider only the act whereby we construct the concept, and abstract from its many determinations (for instance,the magnitude of the sides and of the angles), which are quite indifferent, as not altering the concept 'triangle'. (A 713-714, B 741-742)

As I see it, Kant is not attempting to provide additional warrant for Euclid's proofs. Kant assumes that Euclid's proofs are quite compelling. His account of ostensive constructions is an appeal to details of Euclid's proof to help explain how such proofs provide the warrant that he assumes they obviously do provide.

Kant's discussion of schemata contributes to his explanation of the warrant provided by Euclid's ostensive constructions. The schema for the concept "triangle" is a rule for producing images or empirical figures that can count as constructed plane triangles. Part of the subtle doctrine of the role of what Kant calls the "productive imagination" in empirical recognition of shapes and figures is his idea that the rules for drawing figures are also the rules for recognising figures presented to us in empirical intuition.

> We cannot think a line without *drawing* it in thought, or a circle without *describing* it. (B 154)

> When, for instance, by apprehension of the manifold of a house I make the empirical intuition of it into a perception, the *necessary unity* of space and of outer sensible intuition in general lies at the basis of my apprehension, and I draw as it were the outline of the house in conformity with this synthetic unity of the manifold in space. (B 162)

According to Kant, the rules for drawing a triangle on a piece of paper, or for pacing off a triangle on a football field, are the same rules we use when we see that a figure before us counts as a plane triangle.

What about Kitcher's second objection? Why isn't the ostensive construction unnecessary because the angle sum theorem is already analytically contained in the schema for the concept "triangle"? Consider, first, the analogous question about the schema for the square figure drawn by Plato's slave boy. Knowing how to draw and recognise plane square figures didn't, immediately, lead the slave boy to recognise that a square of twice the area will have a side equal to the diagonal of the original. It was only after first constructing the three additional squares, and then the four connected diagonals, that the way was clear to directly recognise that the square formed by the diagonals is exactly twice the area of the original. The creative construction required to generate the diagram, in which the result could be immediately recognised, shows that the result is not a trivial analytic consequence of the schema for constructing and recognizing plane square figures.

Similarly, the need for creative constructions, such as Euclid's extending the base and his constructing of a parallel to the opposite side which divides the external angle, shows that the angle sum property is not a trivial analytic consequence of the schema for recognizing and constructing plane triangles. The construction leads, by steps each of which could be carried out on any figure that could be a constructed plane triangle, to a figure in which a straight angle is exactly divided by three angles, which have been shown to equal the three angles of the original triangle.

The role of the final constructed figure, in what counts as direct witnessing of the angle sum equalling a straight angle, is analogous to measurement. Ernst Mach provides an interesting operational demonstration of the angle sum theorem that emphasizes the connection between geometry and empirical measurement procedures.

> If a draughtsman draw a triangle by successively turning his ruler round the interior angles, always in the same direction (Fig. 6), he will find on reaching the first side again that if the ruler lay toward the outside of the triangle on starting, it will now lie toward the inside. In this procedure the ruler has swept out the interior angles

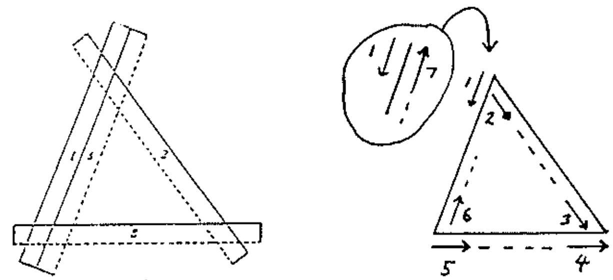


Fig. 6.

> of the triangle in the same direction, and in doing so has performed half a revolution. (1960, 57 arrow diagram added)

For somewhat larger triangles we can think of rotating and sliding an arrow or pointer according to the correspondingly numbered operations. At the end of the operation of sweeping out the interior angles of the triangle it will be exactly reversed in direction. This measures the sum of the angles to be 180 degrees.

Euclid's careful restriction to features unavoidably generable for any construction of a plane triangle is part of a tradition of searching for rigor in mathematical demonstrations that eventually led to conceptions of logical proof that do not depend on ostensive constructions of geometrical figures at all. Mach regards this tradition as having the unfortunate consequence of divorcing geometry from its proper roots in empirical measurement operations. (1960, 112-114) Mach's measurement operation, just as Euclid's constructive proof, invites our confidence in the expectation that we would get the same outcome on any figure we could recognise to be a plane triangle.

As Friedman has argued (1985 op. cit.), Kant's notion of what counts as an analytic consequence of the concept plane triangle was unavoidably limited by the logic available to him. When we consider that Euclid's geometry is three dimensional, so that plane geometry holds only for figures on what can count as flat two dimensional surfaces, there may well be reason to suppose that the angle sum theorem follows logically [according to modern predicate logic] from axioms sufficient to characterize flat surfaces and the definition of a plane triangle. Does this make it analytic that the angle sum theorem holds for any figure we can recognise to be a plane triangle? Whether or not this is so for geometry of what can count as a flat plane, can it be analytic that the space in which we observe and manipulate objects satisfies Euclid's three dimensional geometry?

5. Ostensive Construction and the bounds of observation.

What can count as ostensive constructions for us are limited to scales and tolerances corresponding to our perceptual competencies.[6] As James Hopkins (1973) pointed out, we cannot visualize a scale drawing in which the relative sizes and distances of two stars are accurately represented.

> A simple principle is involved. If a picture is to be taken in, the elements (for example, dots, lines) which compose it must be simultaneously visible. They will therefore have certain spatial properties and relations. Scale pictures like geometric diagrams show spatial situations by the spatial characteristics of their elements. Those characteristics required by considerations of scale may conflict with those needed for visibility. A distortion results from the sacrifice of scale to visibility. (Hopkins 1973, 25)

As Reichenbach (1958, 46) and many others have pointed out, we cannot visually distinguish a right angle from one of e.g. 89 degrees 59 minutes. In a classic paper, Charles Parsons (1964, reprinted in Wolf 1967) pointed out a conflict between the ability to infinitely iterate the bisection construction and the idea that diagrams in what count as ostensive constructions be possible objects of experience.

One of Kant's examples of a claim warranted by ostensive construction is that a straight line is the unique shortest distance between two points (e.g. A 24; B 16; A 163, B 204; A 220, B 268). This is a uniqueness appealed to in Euclid when, in the proof of proposition I.4, it is claimed that two straight lines cannot enclose a space.[7] Hopkins considers attempting to draw diagrams designed to show that two distant points could not be connected by two paths each of which is equally short and both of which are shorter than any other paths.[8]

> Owing to the distortion required to make the lines visible, the only way to make two lines visible would be to bend one away from the other. Then one line would be and appear curved. Hence the only usable (visible) pictures fail to show two lines, or show one curved. (Hopkins 1973, 33)

> At this scale and in this case we can only disregard our images; we cannot take them as showing how things are. So despite the impression, our images are not really Euclidean; rather they are two crude to serve. (ibid, 33)

As Hopkins argues, our inability to draw or visualize such an image does not show that there cannot be more than one shortest path connecting these points.

The same holds for Euclid's ostensive proof of the angle sum theorem. According to Kant, this angle sum theorem applies to any figure we could apprehend to be a plane triangle because the rules for recognizing plane triangles are the same rules that underwrite the ostensive constructions in Euclid's proof. But the tolerances to which this applies to figures we can meet in space are limited by what I have called our perceptual competencies. Any figure too large for us to survey will not be covered.

The need for this sort of limitation on scale can be illustrated by extending Mach's empirical measurement proof to very large triangles on the surface of the earth. Suppose you are at the north pole with a long pointer directed down the 180th meridian (the one of the international dateline). Now turn the pointer 90 degrees counter-clockwise so that it now points down the 90th meridian. Now carry it right down the 90th meridian, keeping it always pointed due south, until the point just reaches the equator. Now swing the butt end 90 degrees counter-clockwise so that it points due west along the equator. Now carry the pointer due west along the equator, keeping the butt end directed west, until the butt end just touches the 180th meridian. Now swing the point 90 degrees counter-clockwise so that it faces north and carry the pointer right up the 180th meridian toward the north pole. When the point reaches the north pole you will find that the pointer has exactly reversed direction. Its point is now at the north pole with its butt end directed due south down the 180th meridian, but you started with the butt end at the north pole and the point facing due south down that meridian. Has your application of Mach's operational demonstration of the angle sum theorem to this large spherical triangle really measured the sum of three 90 degree angles to be 180 degrees?

6. Geometry and Kant's Empirical Realism

Kant believed that his appeal to *a priori* Euclidean constraints on space made possible his empirical realism, thereby preventing his transcendental idealism from collapsing into what he regarded as Berkeley's objectionable form of idealism.[9] Two aspects of Euclidean geometry are relevant to its role in underwriting Kant's empirical realism. First, the three dimensionality of space, together with the theorems connecting three dimensional shape with appearances that would be apprehended from alternative perspectives, is what makes it possible to demonstratively refer to unapprehended appearances that are included in the empirical content that the object of an outer empirical intuition must satisfy. Second, even judgements about appearances make commitments about what would be apprehended from other perspectives. Such judgements are not incorrigible.

These allow inferences from such judgements to expectations about further judgements which are based on more than the sort of Humean inductions of which, according to Kant:

> We can properly only say, therefore, that, so far as we have hitherto observed, there is no exception to this or that rule. (B 4)

If the judgements are taken as incorrigible subjective episodes then expectations about future episodes cannot be based on more than habitual extensions of past regularities. Kant's refutation of idealism argues that the actual sequence of past subjective episodes is no more incorrigibly accessible than our corrigible judgments about outer things. Unless the sort of objective reference to additional appearances made possible by the geometrical constraints on shape and perspective are made available, there is nothing to count as remembering anything about the actual sequence of past episodes. This would collapse the sequence of experiences into a solipsism of the present moment.

II. Riemann on Space and the Foundations of Geometry.

1. Riemann's plan of investigation

On June 10 1854, Riemann gave his Inaugural Lecture "On the Hypotheses which lie at the foundation of Geometry". Here is the plan of investigation which opens Riemann's lecture and motivates his investigation of what he calls "multiply extended quantities".

> As is well known, geometry presupposes the concept of space, as well as assuming the basic principles for constructions in space. It gives only nominal definitions of these things, while their essential specifications appear in the form of axioms. The relationship between these presuppositions [the concept of space, and the basic properties of space] is left in the dark; we do not see whether, or to what extent, any connection between them is necessary, or *a priori* whether any connection between them is even possible.
>
> From Euclid to the most famous of the modern reformers of geometry, Legendre, this darkness has been dispelled neither by the mathematicians nor by the philosophers who have concerned themselves with it. This is undoubtedly because the general concept of multiply extended quantities, which includes spatial quantities, remains completely unexplored. I have therefore first set myself the task of constructing the concept of a multiply extended quantity from general notions of quantity. It will be shown that a multiply extended quantity is susceptible of various metric relations, so that Space constitutes only a special case of a triply extended quantity. From this however it is a necessary consequence that the theorems of geometry cannot be deduced from general notions of quantity, but that those properties which distinguish space from other conceivable triply extended quantities can only be deduced from experience. Thus arises the problem of seeking out the simplest data from which the metric relations of Space can be determined, a problem which by its very nature is not completely determined, for there may be several systems of simple data which suffice to determine the metric relations of space; for the present purposes, the most important system is that laid down as a foundation of geometry by Euclid. These data are—like all data—not logically necessary, but only of empirical certainty, they are hypotheses; one can therefore investigate their likelihood, which is certainly very great within the bounds of observation, and afterwards decide upon the legitimacy of extending them beyond the bounds of observation, both in the direction of the immeasurably large, and in the direction of the immeasurably small. (Spivak, 135-6)

For Riemann, Euclidean geometry is one among other conceivable triply extended quantities that might count as candidates for representing the metrical relations of Space. His concept of a multiply extended quantity is the source of our present concept of a differentiable manifold. It provides the resources to represent whole families of alternatives to Euclidean geometry. If what Riemann refers to as Euclid's system of data for determining the metrical relations of Space are the postulates of Euclid's *Elements*, then he is claiming that Proposition 32 is contingent on empirical hypotheses. He holds that, while the likelihood of these hypotheses is certainly very great within the bounds of observation, the legitimacy of extending them to the very large or very small is to be decided by later investigation.

What Riemann refers to by "the bounds of observation" may well be the same as the limits our perceptual competencies put on Kant's account of how ostensive constructions underwrite Euclid's propositions. Extension to the very large or very small is, at best, by reasoning analogous to what Kant says about ideas of reason. I have earlier argued that the sort of *a priori* status appropriate to such an idea is subject to challenge by producing an alternative that can seriously rival the original's capacity for organizing experience. (Harper 1986) On this view of Kant's philosophy of geometry, Riemann, rather than in diametric opposition, is providing valuable conceptual resources for making application of geometry to the very large and small less dependent on accidental poverty of thought.[10]

Riemann's rather informal presentation does not explicitly show that he had reached the conception of differentiable manifold that grew out of efforts of later mathematicians to develop his work. According to Spivak, however, it is obvious that Riemann was clear on the notion and its basic property that n-dimensional manifolds are locally like n-dimensional Euclidean space.

> However, it is quite obvious that the notion was thoroughly clear in his own mind and that he recognized that manifolds were characterized by the fact that they are locally like n-dimensional Euclidian space. (Spivak, 155)

An n-dimensional space is locally Euclidean if in a neighbourhood of any point it admits a metric homeomorphic to the standard Euclidean distance metric ($d(x,y) = [\Sigma_i(x_i-y_i)^2]^{1/2}$) in $\mathbb{R}^n$. This metric on $\mathbb{R}^3$ is exactly the distance relation between points construed as triples of real numbers that corresponds to three dimensional Euclidean geometry. On Riemann's internal characterization of n-dimensional curvature a region of a manifold counts as flat—having zero curvature—just in case the distance between any pair of points in it satisfies this Euclidean metric. (Spivak, 143)

While Riemannian manifolds are compatible with the idea that what counts as locally Euclidean holds up to the bounds of observation they do not require it.[11] Riemannian manifolds can represent spaces that are quite non-Euclidean at scales corresponding to what we can optimally discriminate by our observations. We shall want to ask to what extent Riemann's suggestion that the likelihood of Euclidean data is very great within the bounds of observation is supported by Kant's account of Euclidean constructions. We shall also want to ask whether such constructions can support anything that could count as more than empirical certainty.

2. Topology

Riemann distinguishes what we would now call topological properties of manifolds from metrical properties. Among such properties are the unboundedness and three dimensionality of Space, which he takes to be known with empirical certainty greater than that of any experience of the external world.

> That Space is an unbounded triply extended manifold is an assumption which is employed for every apprehension of the external world, by which at every moment the domain of actual perception is supplemented, and by which the possible locations of a sought for object are constructed; and in these applications it is continually confirmed. The unboundedness of space consequently has a greater empirical certainty than any experience of the external world. (Spivak, 150)

The focus on unboundedness reflects the role of this passage as part of a larger discussion pointing out that our certainty about unboundedness is compatible with our uncertainty about the infinitude of space. (150-151) One may presume that, for Riemann, the claim that space is triply extended shares the same extreme empirical certainty as the claim that it is unbounded.

A single topology is compatible with a variety of different metrical geometries. Torretti suggests that the distinction between topology and metric can be exploited to inform an interpretation of Kant which would allow for alternative metrical geometries.

> Since Kant conceived the "manifold of a priori intuition" called space, not as a mere point-set, but as a (presumably three-dimensional) continuum, we must suppose that he would have expected "the mere form of intuition" to constrain the understanding to bestow a definite topological structure on the object of geometry. But, apart from this, the understanding may freely determine it, subject to no other laws than its own. Since the propositions of classical geometry are not logically necessary, nothing can prevent the understanding from developing a variety of alternative geometries (compatible with the prescribed topology), and using them in physics. (Torretti 1984, 33)

On this suggestion topological properties, such as continuity, three dimensionality, orientability and unboundedness, count as constraints directly imposed by the mere form of intuition, while metrical constraints, such as the angle sum theorem, do not.

One difficulty with this suggestion is that global topology is not obviously more accessible than global metric geometry. For example, black holes would generate discontinuities in space-time thus distorting topology. (Earman 1995; Wald 1984, 148ff) Should event horizons associated with such singularities be counted as boundaries? Whatever we say about this, we do not want to rule out singularities by demanding that only manifolds without discontinuities could count as models of space-time. Just as the notion of differentiable manifold developed from Riemann's lecture provides for manifolds with different metric geometries at different locations so does it provide for manifolds with quite radically differing local topologies at separated submanifolds. (see e.g. Spivak Vol.1, p. 4)

One advantage Riemann points out for topological features such as three dimensionality is the discreteness of the set of alternatives.

> In this connection there is an essential difference between mere relations of extension and metric relations, in that among the first, where the possible cases form a discrete manifold, the declarations of experience are to [be] sure never completely certain, but they are not inexact, while for the second, where the possible cases form a continuous manifold, every determination from experience always remains inexact—be the probability ever so great that it is nearly exact. (Spivak, 150 [be] added)

Measuring curvature, like measurement in general, does not fix exact values, however great the precision to which nearness to an exact value is established. Thus, as Torretti (104-105) points out, the extent to which space could be known to have the zero-curvature characterizing Euclidean metric geometry is limited by measurement tolerances. Such tolerances could allow ruling out, with very high probability, curvatures outside an interval (-e, +e) for some positive bound e, but they would not allow us to establish that the curvature is exactly zero, even locally.

Kant included the three-dimensionality of space among the propositions he thought could be established with certainty about space. (e.g. B 41) The following remark from B 154 immediately follows the passage about drawing figures cited in the discussion of productive imagination above.

> We cannot represent the three dimensions of space save by *setting* three lines at right angles to one another from the same point.

It suggests that Kant would regard our certainty of the three dimensionality of space as represented by our certainty that three is the maximum number of lines that we could set to meet at right angles at a single point. Our interpretation of the role of schemata, as rules for constructing or recognising configurations in the space in which we move our bodies and manipulate objects, suggests that Kant interprets ostensive constructions to represent commitment to the same sort of general features of experience alluded to by Riemann.[12]

On this interpretation, the sort of very general appeal to experience Riemann alludes to as warranting the three dimensionality and unboundedness of space may also count as informing the warrant provided by ostensive constructions underwriting claims about metrical geometry, once those claims are appropriately limited in scale and tolerances to correspond to our perceptual competencies. I suggest, therefore, that limiting what ostensive constructions are supposed to warrant to such scales and tolerances is more promising than trying to limit what the form of intuition can deliver on to abstract topology rather than metric geometry.

3. Movable Rigid Bodies

Metric geometries of constant curvature are salient among the examples of manifolds discussed by Riemann.

> The common character of those manifolds whose curvature is constant may be expressed as follows: figures can be moved in them without stretching. (Spivak, 147)[13]

Riemann suggests that the empirical notions on which the metric determinations of Space are based are the concept of a solid body and that of a light ray. (Spivak, 152) For Helmholtz, the requirement that empirical measuring operations, corresponding to moving rigid rods about in space, be provided for counted as a justification for limiting metric geometries to those of constant curvature. (EW, 56-57)[14] Helmholtz points out that adding the further condition that shape be independent of size restricts these metric geometries to the unique Euclidean case of zero curvature. (PSL, 239)

Like the constraints on topology, scale considerations are relevant here. When we consider the familiar experiences of empirical measurement on which activities such as building ships, houses and furniture are based we do see a central role played by moving rods that we count as rigid. At the scales and tolerances involved, however, we also see uncontroversial appeal to the full Euclidean metric, up to the relevant tolerances. For example, in the use of scale models in designing hull shapes of ships.

When we consider astronomical distances, even limited to those exhibited in the solar system phenomena that provided the main evidence for Universal Gravitation, the moving of rigid rods plays no direct role at all. It certainly was never plausible to demand that any geometry that could count as a candidate for the metric of physical space should provide resources to allow for the conceptual possibility of measuring distances between planets with giant measuring rods. This demand is not made plausible, even if what counted as edges of such rods were allowed to follow geodesics of geometries of constant curvature. Was it ever reasonable to demand the conceptual possibility of, even, the idea of laying a small measuring rod over and over again along a shortest path (a light ray or an ideally stretched string perhaps?) between planets?[15]

Considerations of the very small make appeal to light rays, as well as to rigid bodies, problematic. This is explicitly discussed by Riemann in the following interesting passage:

> Now it seems that the empirical notions on which the metric determinations of space are based, the concept of a solid body and that of a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypotheses of geometry; and in fact one ought to assume this as soon as it permits a simpler way of explaining phenomena. (Spivak, 152)

In addition to testifying that Riemann takes the concepts of solid body and light ray to be the empirical notions on which the metric determinations of Space are based, this remark shows extraordinary good methodological sense. At very small, submicroscopic scales, there is even less warrant for imposing geometrical properties as *a priori* commitments on physical investigations.

Another of the few examples Riemann provides is one in which astronomical measurements tell us that the total curvature of every measurable region is not perceptibly different from zero; but, even so, at every point the curvature can have arbitrary values in each direction.

> —; at every point the curvature can have arbitrary values in three directions, providing only that the total curvature of every portion of Space is not perceptibly different from zero.

At more limited scales, and restricted to two dimensions, this sort of example covers all of what count as flat surfaces we can meet with in our space.[16] Floors, walls, table tops, even the most polished flat mirror, all count as bumpy at small enough scales, even though at scales corresponding respectively to appropriate tests of their flatness the average curvature does not count as perceptibly different from zero. For a table top, the appropriate criterion of flatness might be to be able to slide a level or straight edge about over the surface with no perceptible rocking or visible gaps between the straight edge and the surface.

III. Reichenbach on Visualization and Conventionality of Geometry.

1. Visualization of Non-Euclidean Geometry.

Reichenbach provides examples designed to show that people could learn to visualize according to non-Euclidean geometries. In these examples there is non-Euclidean topology as well as metric geometry. The first example is a story of a man exploring and measuring spheres which exhibit topological relations corresponding to three dimensional analogues of non-intersecting closed curves on a torus surface. He argues that to retain Euclidean geometry the man would have to accept a duplication of happenings that would require, in addition to distorting forces, a causal anomaly. (1958, 65-67)[17]

Reichenbach does not attempt to give a detailed account of visualization of the torus space. Within each shell, the standard Euclidean coordinative definitions are employed to interpret measurements. His attempt at a detailed account is, instead, for a three dimensional spherical geometry that exhibits, for three dimensional spherical figures, the relativity of enclosure of circles on a two-dimensional spherical surface. Think of two circles of latitude on a sphere, circle I is south of the equator, the other, circle II, is equally far north of the equator. What Reichenbach calls stereographic projection from the north pole to a plane tangent to the south pole will make the projection of II completely enclose the projection of I. (1958, 69-72) An alternative projection, from the south pole to a plane tangent to the north pole, would reverse this enclosure. According to Reichenbach, this relativity of enclosure would be exhibited by spherical surfaces in a space of three dimensional spherical geometry. (1958, 71-73) He uses the assumption that light rays travel in geodesics (shortest paths) of this geometry to make drawings representing how this relativity of enclosure could be visually experienced by someone exploring an appropriately sized space of this sort.

> Let us imagine in space two large spherical shells I and II, made of sheet metal, which enclose each other and are rigidly connected by beams. An observer climbs around between the shells; however, he cannot pass through them but is restricted to the space between the spheres. He intends to determine which shell is the outside one.
> In order to visualize his experience we construct the following figure. In the stereographic projection we draw two concentric spheres I and II, the top view of which can be seen in Fig. 10. (1958, 71)

> The observer is stationed at *A* on the fundamental circle. In order to make our problem precise we assume that light rays move along straightest lines in space. Hence we can determine by means of main circles and main spheres what is visualized by the observer, just as in Euclidean space a representation of his perception is ascertained

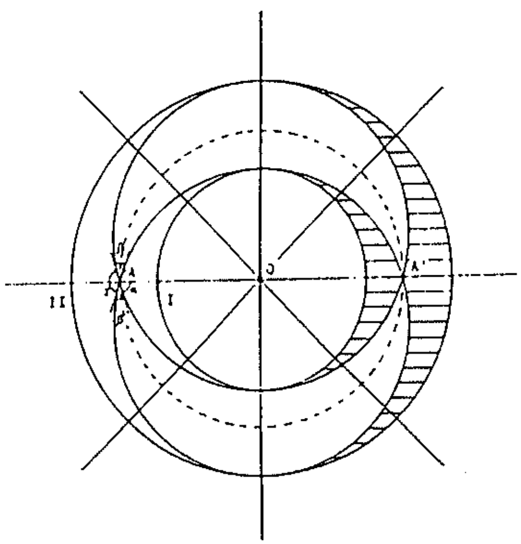
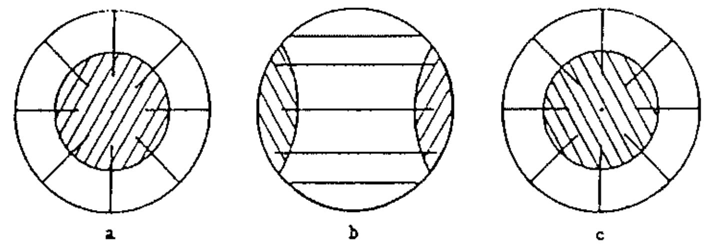


Fig. 10. Stereographic projection of spherical space: perspectives of an observer at *A*.

> by means of straight lines and planes. We draw two main circles through *A* which are tangential to the two circles I and II; they furnish the angular perspective for *A* as do the corresponding lines of projection in Euclidean space. Since in every plane through *A* and *O* the same relations hold, we may conceive Fig. 10 as a cross-section through a three-dimensional figure which results from a rotation around the axis *AOA'*. If the view of the observer is confined to the angular space α he will see shell I; in the angular space β [γ] he will see shell II, and in the angular space γ [β] he will see the empty space between the shells .1958, 72-73)[18]

Reichenbach goes on to give pictures corresponding to the observer's perceptions in the directions of these views.

> For the purpose of constructing the picture of his perceptions we must follow the cone of light rays which begins at *A*; since the stereographic projection reproduces the original angles, this cone is immediately given by the tangents at *A*. It is a double cone symmetrical to the line *AO*. His perceptions are obtained by the intersection of this double cone with a plane of projection, which must always be assumed to lie perpendicular to the direction of view. Fig. 12 shows the views which are seen in the three directions: (a) from *A* towards I, i.e., along the central axis of α; (b) from *A* along a direction perpendicular to the first, i.e., along the central axis of the adjacent angular space β; (c) from *A* towards II, i.e., along the central axis of γ. In these figures the shells are distinguished from each other by different shadings; Fig. 12*a* shows the shading of shell I, Fig. 12*c* that of shell II.



> We can now imagine the visual experiences of the observer. From the space between the shells he sees both spheres as *convex* surfaces; i.e., by looking towards the spheres he discovers that light rays do not glide along the surface, and that within the space between the shells there is no connecting light path between two points on the same surface. If he stands in the middle of the shell space looking toward I, he sees in front of him the convex hemisphere of this surface surrounded by free space; when he turns around, he sees shell II in the same manner, i.e., its convex hemisphere surrounded by free space. (1958, 74-75)

The symmetry of these views suggests that the space generated from the opposite stereographic projection—where the south pole would be the origin for projection onto a plane tangent to the north pole and circle I would enclose circle II—would not present visual images that could be distinguished from these.[19]

Reichenbach goes on to explore perspectives corresponding to an eye right next to a shell and to explore the visual experiences that would be generated by being able to open holes in the shells that could be looked through and climbed through to produce additional perspectives. He argues that these perspectives are also unable to provide information that specifies which shell encloses the other. According to his description, going into what appears to be the inside of either would produce views of the other, through the holes, that would make it appear that the other shell encloses the one the observer has climbed into. (1958, 76-77) This is a very clever illustration of visual perceptions that would exhibit the non-Euclidean relativity of enclosure of spherical surfaces in a three-dimensional geometry corresponding to what Reichenbach describes as spherical space.[20]

Reichenbach cites Helmholtz as his authority for taking specification of appropriate sequences of such "perceptions" as what would count as visualization corresponding to the non-Euclidean spherical space.

> We shall follow Helmholtz's method; for "visualize" Helmholtz gave the definition: "...that we are able to imagine the series of perceptions we would have if something like it occurred in an individual case." (1958, 63)

In introducing his mapping exercise Reichenbach suggests that to imagine the sequence of such perceptions it is sufficient to draw two-dimensional plane figures corresponding to retinal images.

> From our drawing we shall be able to infer his perspective, which is very different from the Euclidean one. We can then draw without distortion pictures that represent his perceptions, because we merely have to draw plane figures which, when projected on the retina, will furnish the same pictures as those actually occurring in spherical space. (1958, 68-69)

After generating such pictures, and pointing out how just as in three-dimensional Euclidean space no single visual image of the whole space is possible, Reichenbach describes how sequences of perceptions can represent spherical space by visual integration.

> It is quite possible, however, to attain a general impression of spherical space by *visual integration* so to speak, i.e., by looking around and pacing it off. (1958, 78)

He specifies that measurements, e.g. by rigid rods, agree with the congruence specified by spherical geometry, though he doesn't give many details. (1958, 76) Reichenbach suggests that, once the perceptions of an observer in such a space have been described, the interesting exercise for epistemology is to start with the assumption that an observer has such perceptions.

> Assume that an observer has the perceptions described above; what would he infer? Only in this form is the problem accessible to an epistemological analysis. So long as we start from a certain state of the universe and infer perceptions from it, an aprioristic philosophy may contend for any number of reasons that such a state of the universe does not exist. As soon as we start from perceptions, however, the objection disappears, because nothing can be prescribed for perceptions. No *a priori* postulate can exclude the possibility that some person may at some time have certain perceptions. Only the *interpretation* of such perceptions is controversial. (1958, 77)

> The interpretation is simple for non-Euclidean geometry. In this interpretation we deal with spherical space; there is no absolute "exterior" for the spheres; each of them is the exterior one with respect to the corresponding point of contraction. The two points of contraction are given in the model by the centers of the scaffolds. (1958, 77)

He goes on to argue that, just as with the torus example, maintaining Euclidean geometry would require not just distorting forces but causal anomalies as well. (1958, 78-79)

Reichenbach wants his possible perceptions to be able to count as something more than illusions or hallucinations.[21] His suggestion that we can assume that the observer has such perceptions independently of assumptions about their interpretation is designed to avoid issues that can be raised about whether or not his examples ought to count as really possible. It is not at all clear that the stories Reichenbach tells do count as real possibilities. For example, if the force producing the visible light bending effects he describes were much like gravity humans would not be able to function in it. In fact it may not be unreasonable to treat the outcomes of Euclidean constructions, when restricted to scales and tolerances corresponding to the bounds of observability, as *a priori* in the sense that Reichenbach's examples can be treated as mere conjectures until such time as actual phenomena require revisions which would allow for such examples.

This sense of "*a priori*" is one Reichenbach, perhaps, could grant, at least for Euclidean topology, if the source of the assumed warrant could be counted as empirical. His target is a notion of *a priori* warrant based on the idea that it is impossible to specify visual images corresponding to perceptions in a three dimensional non-Euclidean geometry that would visibly differ from Euclidean images. He takes Kant's thesis to be that the force of Euclidean visualization is provided by the picture or diagram itself. (1958, 38) His clever examples do show that two dimensional images corresponding to visual perspectives of visibly non-Euclidean spaces can be specified.

On Kant's account, the warrant provided by ostensive constructions is provided not just by the diagrams themselves but also by their interpretation as carrying information about the space in which we move about and manipulate approximately rigid bodies. To treat a specified sequence of "perceptions" as evidence for non-Euclidean geometry requires commitment to observer independent generalizations that can count as phenomena exhibiting the systematic dependencies underwriting the interpretation. It is not obvious that specification of two-dimensional pictures that would correspond to retinal images produced from various perspectives which might be met when visually exploring such a space is sufficient to characterize what should count as non-Euclidean visualization.

In his comments on Helmholtz's "On the facts underlying geometry" Paul Hertz makes the following contrast between the intuitions underwriting Euclid's axioms and the successive representations required by Helmholtz.

> On the other hand, Euclid's axioms have the advantage that their truth is evident immediately in the intuition of the simultaneously given, whereas with Helmholtz's axioms even understanding is not possible other than in successive representations. Thus they can compel our assent at most on an associative basis. (EW, 61)

Very likely Hertz had in mind the sort of visualization that apparently forces consent to the uniqueness of the straight line connecting two points according to Euclid's first postulate. As our discussion of Hopkins made clear, such visualization does not justify such uniqueness at scales and tolerances exceeding our perceptual competencies. At appropriately limited scales and tolerances, however, the ability to survey the entire figure from a single perspective may well contribute to the warrant such visualization provides. In the angle-sum construction, at scales consistent with our optimal perceptual competencies, the entire figure can be surveyed from a single perspective. This differentiates the sort of successive representation appealed to by Kant's account of ostensive constructions from Helmholtz's account of visualization.[22]

Hertz suggests that Helmholtz's limitation to merely successive representation in his account of visualization makes it able to compel our assent at most on an associative basis. This suggests that it would be limited to what Kant takes to be the "merely assumed and comparative universality" conferred by experience through induction. Kant's appeal to what Reichenbach calls Euclidean congruence makes his perceptual judgments about rigid bodies carry commitments about what would be apprehended from other perspectives and by touch and manipulation. These information-carrying commitments would require regarding earlier judgments as having been in error if conflicting information were later to be obtained from such further explorations.

Helmholtz also interprets perceptual judgments as making such commitments. He assumes such commitments to result from a history of perceptual associations, even though the relevant sequence of experiences may not be available to conscious memory.

> In previous studies I characterised as *unconscious inferences* the connexions between representations which thereby occur—unconscious, inasmuch as their major premise is formed from a series of experiences, each of which has long disappeared from our memory and also did not necessarily enter our consciousness formulated in words as a sentence, but only in the form of an observation of the senses. (EW, 132)

In his comments Schlick quotes the following remark from Helmoltz's *Physiological Optics*

> Thus although a genuine conscious inference is not present in these cases, the essential and primary task of one is accomplished and its result achieved, if admittedly only through the unconscious association of representations. This association goes on in the obscure background of our memory, and its results also force themselves upon our consciousness as if obtained by way of a compelling, seemingly external power, over which our will has no authority. (EW, 175-176)

As Joan Richards (1977, 239) has pointed out, Helmholtz's appeal to such unconscious inferences makes the differences between his empiricism and the nativism he opposed more subtle than one might have expected.

The nativists argued that what Helmholtz calls unconscious inferences based on a long history of empirical associations are actually based on more or less wired in features of our sensory systems. If some such nativist doctrine were correct then the plasticity of geometric interpretation assumed by Reichenbach might not be available. This might support treating propositions proved by appeal to Euclidean ostensive constructions, when appropriately limited to bounds of observability, as *a priori* commitments in a very strong sense according to which anything that failed to meet them could not count as a possible object of experience for us.[23]

Though his important *Handbook of physiological optics* was very influential in getting a wide audience for his attacks on nativism, it did not lead to a lasting consensus in favour of his empiricism. In the 1880's the literature in perception slowly began to swing toward nativism. (Turner 1993, 198) One factor promoting this trend was the growing acceptance of evolutionary theory, which made it attractive to regard a spatial sense as based on physiological developments occurring over the life of the species rather than merely on associations occurring anew during the first few months in the life of each individual. (Turner, 198) Issues over nativism continue to be debated without any conclusive resolution to our present day.[24]

On Helmholtz's empiricist account, as well as on Nativist accounts, the warrant provided to Plato's slave boy by his ostensive construction, the very first time he performs it, is far more compelling than would be provided by a single trial on an associationist account of induction. Whether, even if we put aside worries about whether they could actually occur, examples like the ones Reichenbach proposes could be expected to provide anything like rich enough sequences of associations to back up what would count, for non-Euclidean geometry, as the sort of unconscious inferences Helmholtz postulates for Euclidean visualization is not obvious.

It is also not obvious that more is not involved in the warrant provided by Euclidean constructions than any set of mere associations, however rich they may be, could provide. Consider attempting to apply Euclid's angle sum construction to the large spherical triangle on the surface of the earth to which we applied Mach's empirical measurement procedure. Let us also, for the present, put aside the requirement that the constructed figure be surveyable from a single perspective. Let the north pole count as the point C and the 90th meridian from B where it crosses the equator to the pole count as the side BC. The remaining corner A will be where the 180th meridian crosses the equator. The first step in Euclid's construction, extending BC to some point D, is not problematic. As we will be attempting to carry out the construction at the north pole let D be put some short distance (perhaps even only a meter or two) beyond the pole on the great circle extending BC. The next step, however, turns out to be impossible, if what count as straight lines are limited to great circles. How do we construct at C, the North Pole, a line parallel to BA, which runs along the equator. Every straight line (great circle) running through C is exactly perpendicular to BA.

If we attempted to use parallels of latitude to construct a small version of our figure where points B' and A' were just short equal distances down the 90th and 180th meridians the side $B'A'$ would become more and more obviously sharply curved as the equal distances CA' and CB' became shorter. To have a figure count as a triangle (segments of great circles as sides) with a 90 degree angle between segments of the 90th and 180th meridians at point C as a vertex, the sum of the angles at the other vertexes $B'$and A' would become more and more constrained toward equalling 90 degrees as the lengths of the sides became shorter. The capacity to construct at C a segment EC of a great circle such that angle $A'CE$ did not differ from angle $B'A'C$ and angle ECD did not differ from angle $A'B'C$ by more than specified tolerances would allow fixing tolerances by which their sum was constrained to approximate 90 degrees. This illustrates how the capacity to carry out Euclid's construction carries information forcing approximations to which the angle sum theorem holds.

One important difference between Euclid's constructive proof and Mach's measurement operation is that the outcome of Mach's measurement operation, which is to be interpreted as representing the angle sum equalling 180 degrees, can be arrived at erroneously unless the surface is independently specified to be flat. Mach's procedure does seem to require the sort of circularity of which Kitcher accused Kant's account of Euclid. As we have just seen, however, Euclid's construction actually provides a test verifying that the surface counts as sufficiently flat. It is like a null experiment establishing bounds limiting violations of the theory for figures at scales corresponding to our perceptual competencies.

2. Conventionality of Geometry.

In his plan of investigation (quoted above) Riemann suggested that there may be several systems of data which suffice to determine the metric relations of space. If Reichenbach's coordinative definitions can count as what Riemann calls "systems of simple data which suffice to determine the metric relations of space", then his thesis that metrical geometry is conventional can be seen as a particularly radical version of Riemann's suggestion that the problem of finding such simple data is not completely determined.

Consider Reichenbach's example of people using measurements to determine the shape of a plane with a hump that they are climbing around on.

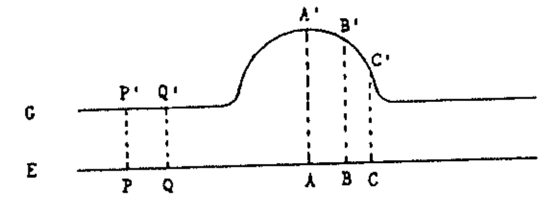


Fig. 2. Projection of a non-Euclidian geometry on a plane.

> Let us imagine (Fig. 2) a big hemisphere made of glass which merges gradually into a huge glass plane; it looks like a surface G consisting of a plane with a hump. Human beings climbing around on this surface would be able to determine its shape by geometrical measurements. They would very soon know that their surface is plane in the outer domains but that it has a hemispherical hump in the middle; they would arrive at this knowledge by noting the differences between their measurements and two-dimensional Euclidean geometry.
>
> Let us assume that the plane E is also inhabited by human beings and let us add another strange assumption. On the plane a mysterious force varies the length of all measuring rods moved about in that plane, so that they are always equal in length to the corresponding shadows projected from the surface G. Not only the measuring rods, however, but all objects, such as all the other measuring instruments and the bodies of the people themselves, are affected in the same way; these people, therefore, cannot directly perceive this change. What kind of measurements would the E-people obtain? In the outer areas of the plane nothing would be changed, since the distance $P'Q'$ would be projected in equal length on PQ. But the middle area which lies below the glass hemisphere would not furnish the usual measurements. Obviously the same results would be obtained as those found in the middle region by the G-people. Assume that the two worlds do not know anything about each other, and that there is no outside observer able to look at the surface E—what would the E-people assert about the shape of their surface?
> They would certainly say the same as the G-people, i.e., that they live on a plane having a hump in the middle. (1958, 11-12)

Given the usual definitions of congruence based on what standardly count as rigid bodies, the E people, just as the G people, would count themselves as having empirically established that their surface is a plane with a hemispherical hump in the middle.

Reichenbach goes on to characterize the sort of undetectable forces making the measurements in the flat plane E agree with those of the humped space G as universal forces—forces which affect all materials in the same way and from which there are no insulating walls. (1958, 13) Later he argues for the relativity of geometry by suggesting that any Riemannian metric geometry can be transformed into an arbitrary alternative Riemannian metric geometry by imagining a universal force. (1958, 33)[25]

> Given a geometry G' to which the measuring instruments conform, we can imagine a universal force F which affects the instruments in such a way that the actual geometry is an arbitrary geometry G, while the observed deviation from G is due to a universal deformation of the measuring instruments.

According to Reichenbach, the usual underlying coordinative definitions of congruence and rigid body, which rule out universal forces (1958, 22), are conventions, much like the choice of meters as units of length. (1958, 35)

Suppose the universal force distorting the measurements of the *E*-people could be fleshed in sufficient detail to count as an example where the metric geometry of a flat plane is transformed into the metric geometry of a plane with a hemispherical hump. Presumably, a corresponding universal force could then be defined which would transform the humped geometry of the *G*-world into a flat plane. Consider *G*-people who recommended adopting such alternative coordinating definitions in order to preserve the geometry of a flat plane. These *G*-advocates of flat geometry would be like people who have actually succeeded in working out detailed alternative coordinative conventions to make a flat earth conjecture compatible with the outcomes of geodetic measurement operations.

According to Reichenbach, these *G*-world flat plane true believers are not making what should be counted as a factual mistake. He assumes that they do manage to appeal to alternate coordinate definitions that really are adequate to a universal force that distorts measurement outcomes on a flat plane so that they exactly agree with what the standard coordinate definitions would take to be measurements of a surface with a hump in the middle. On this assumption, their assertion that the surface is a flat plane with a force distorting measurements in the middle would make exactly the same predictions about what congruences would be obtained between measuring rods and parts of the surface at any location and orientation. In so far as empirical data are limited to such operationally defined measurement outcomes, and claims are regarded as empirically equivalent if they make exactly the same predictions about these empirical data, the claims of the flat plane advocates are empirically equivalent to those of the round hump advocates.

Even if we grant these assumptions, we need not be committed to grant that the flat plane conjecture counts as equally supported as the hump conjecture by the measurement outcomes which they both predict. In the hump conjecture the standard definition of rigid body does without the assumption of distorting forces making the coordinative definitions applying in the central region differ from those applying at other locations. This uniformity, over different regions, of the force laws corresponding to the standard coordinative definitions is an advantage. As far I can see, it is only Reichenbach's operationalism which prevents him from counting this as a cognitive or epistemological advantage.[26]

According to Reichenbach (1958, 12-13, 22), on the standard definition of rigid bodies the only forces that can be appealed to as distorting bodies are ones that can be independently measured by what he calls differential effects. This also counts as an advantage. Newton's methodology appeals to an ideal of empirical success according to which a successful theory not only predicts accurately, but, also, has its theoretical parameters accurately measured by the

phenomena it purports to explain. (Harper, forthcoming) On such a criterion the advantage of minimizing claims about forces that are not backed up by independent measurements counts as a direct contribution to empirical success.[27] The theory according to which the hump is there does better on Newton's criterion of empirical success than the theory of the *G*-world flat planers which requires appeal to such undetectable forces.

These considerations suggest that, even if the *G*-world flat planers could deliver on alternative coordinating definitions, the empirical evidence would support the standard theory better than their alternative hypothesis. Consider, now, the assumption that the *G*-world flat planers could succeed in formulating alternative coordinating definitions that correspond to appropriate distorting forces. By focusing on measurements with rods Reichenbach suggests that such forces could be specified by just specifying how the lengths of the rods depend on position and orientation. This focus on measurement, operationally defined by specifying congruence relations, may have seemed to increase the plausibility that transformations made available by Riemann's definition of metric could count as an adequate specification of such forces.

When other indicators of humped shape are considered it becomes even less plausible that a transformation in differential geometry will count as enough to adequately specify appropriate deforming forces. For example, presumably, gravity also needs to be appropriately distorted. In the *G*-world balls would role down the hump, while in the *E*-world they would not roll unless pushed. Forces adequate to compose with *E*-world gravity, which is everywhere normal to the surface, to produce total force agreeing with *G*-world gravity would produce tidal effects proportional to the curvature at locations on the hump. Additional forces would be needed to offset these tidal effects. In the *G*-world, I presume, people out on the flat part could see the hump rising up in front of them. Additional forces would be required to bend light rays so that an *E*-world would appear to have a hump.

Scale considerations also apply here. Suppose the hump on the *G*-surface is a hemisphere of only about ten feet in radius. Here the curvature of visible parts of the surface would be accessible to visual survey. Helmoltz's discussion of looking at reflections in a convex spherical mirror suggests, in rough general terms, how mappings definable with the resources of Riemann's differentiable manifolds could represent systematic distortions transforming any given retinal image of a flat *E* plane into a corresponding image of the *G* hump from the given perspective. (PSL., 241-243) But, distorting forces which could produce such effects, and also mimic the proper coordination of these with motor control and proprioceptive sensory information about eye and head movement which normally make a sequence of perceptions carry information about a humped shape, are not at all plausibly defined by the resources of differential geometry.[28] When, in addition, we include sensory and motor perceptions involved in approaching to handle and apply instruments to the surface, such complications are greatly increased.[29]

The problem is finding alternative coordinative definitions that transform a sequence of perceptions into ones which, instead of accurately carrying information about a hump, accurately carry information about the orientation and location relative to the observer of surveyed parts of a flat surface. If the resources of differential geometry were adequate to define the flat plane as an alternative to the hump then they would be adequate to define additional alternatives, such as differently sized humps and other shapes. This undercuts the very idea that perception can carry information about shape. If it were extended to shapes of moderately rigid objects we can manipulate manually it is hard to see that there would be enough left of Reichenbach's idea of directly observed measurement outcome to distinguish his operationalism from an objectionably radical sceptical relativism.[30]

IV. Concluding Remarks

Reichenbach's own discussions (1958, 37-47) of Euclidean visualization, and especially the discussions of Helmholtz which may have inspired them (e.g. EW, 41; PSL, 226, 240, 245-246) suggest that the key to the force of the usual Euclidean congruence in our ordinary experience is the extent to which it carries information about shapes and relative positions of what count as the many approximately rigid objects which we observe and handle. Regularities coordinating sight with touch and motor information, touch with touch and manipulation, as well as those coordinating sight with perspective are included in what count as phenomena underwriting the normative force of Euclidean congruence, at the scales and tolerances corresponding to our perceptual competencies.[31]

It may be worth calling attention to the role of phenomena in Riemann's proposal for how an empirical answer to the question of determining geometrical structure can be found.

> An answer to these questions can be found only by starting from that conception of phenomena which has hitherto been approved by experience, for which Newton laid the foundation, and gradually modifying it under the compulsion of facts which cannot be explained by it. Investigations like the one just made, which begin from general concepts, can serve only to insure that this work is not hindered by too restricted concepts, and that progress in comprehending the connection of things is not obstructed by traditional prejudices. (Spivak, 152-153)

The core of Newton's conception is a sharp distinction between propositions gathered from phenomena by induction and merely conjectured hypotheses, as expressed in his 4th Rule for reasoning in Natural Philosophy.

> In experimental philosophy, propositions gathered from phenomena by induction should be considered either exactly or very nearly true notwithstanding any contrary hypotheses, until yet other phenomena make such propositions either more exact or liable to exceptions. (Cohen and Whitman, 388)

For Newton the heart of what counts as gathering propositions from phenomena by induction is that the propositions can count as assignments of parameter values that are measured by those phenomena.[32]

On this rule the deliverances of the standard Euclidean congruencies at the scales and tolerances corresponding to our perceptual competencies would be maintained unless other phenomena forced revisions. A rival congruence would count as a mere hypothesis, which ought to be ignored, unless it could realize Newton's ideal of empirical success comparably well. As we pointed out above, the *G*-world flat-plane congruence is clearly inferior to the standard congruence on this criterion of empirical success.

Commitment to the standard Euclidean congruence at the scales and tolerances corresponding to what Riemann calls the bounds of observation allows the massive array of approximate measurement data provided by rigid rods to count as maintained knowledge, even when the extension of our investigations to the very large leads us to alternative metrical geometries.[33] On this way of construing Kant's empirical realism science is a natural extension to less directly accessible phenomena of the same ideal of empirical success that underwrites our everyday practical measuring operations.

Notes

* This paper benefited from criticism of a late draft by Robert DiSalle, Lorn Falkenstein, Curtis Wilson, and Clark Glymour, and from consultation with David Malament, John Earman, Michael Friedman, Carl Posy, and Kenneth Manders. None of them is responsible for any of my errors of content or emphasis.

1. Passages from Kant's *Critique of Pure Reason* are referred to by A and B page numbers, translations are from Kemp Smith.

Emily Carson has presented excellent discussions of Kant on intuition in mathematics and Kant on definition in mathematics which bring out positive aspects of Kant's account of ostensive constructions and can add to what I discuss here.

2. Euclid does not bother explicitly citing any justification. But, this construction is an immediate application of his postulate 2—to produce a finite straight line continuously in a straight line. (Heath, 154 with discussion on 196-199)

3. Proposition I.31—Through a given point to draw a straight line parallel to a given straight line. (Heath, 315-316)

4. Proposition I.29—A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles. (Heath, 311-314)

5. Proposition I.13: If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles. (Heath, 275-276)

6. I use "perceptual competencies", rather than "perceptual capacities" (as Hopkins does), in order to suggest an analogy between knowing how to make perceptual judgments, at scales and tolerances where our ability to discriminate is optimal, and the sort of knowledge of grammatical correctness that makes a person count as a fluent speaker of a language. My point is to emphasise the normativeness of Euclidean constructions for perceptual judgments about what count as the many approximately rigid bodies we observe and manipulate. I will be arguing that such norms are grounded in aspects of our

form of life that make them far less conventional than differences between British and American English usage or the adoption of metric units for measuring.

7. This uniqueness is said to be implicit in Euclid's first postulate.

> Let the following be postulated: to draw a straight line from any point to any point.

See Heath, 195-196 for discussion and 248 for Euclid's appeal (in the proof of proposition I.4) to the impossibility of having two straight lines enclose a space.

8. Reichenbach (1958, 32) uses our "seeing" that a straight line we have drawn connecting two points is shorter than a curved line connecting them as an example of visualization of Euclidean geometry.

> The power of imagination compelling us to make this assertion is called the ability of *visualization*.

As we shall see, Reichenbach argues that the normative basis for such visualization is a convention—the notion of rigid body built into our coordinative definition of congruence.

9. According to Kant, Berkeley is unable to provide for an adequate distinction between truth and illusion, because he attempts to make space a mere empirical representation (*Prolegomena*, appendix translation in Ellington 1985, 114). These issues are discussed in more detail in Harper. (1984, reprinted in Posy 1992)

10. It may be worth pointing out that Riemann, like Kant, did not have available modern predicate logic. Riemann's ability to appeal to resources of differential equations to construct his notion of a manifold suggests that Friedman's emphasis on the idea that without modern predicate logic there was no way to conceptualize alternatives to Euclidean geometry needs some qualification.

11. The explicitly required sense of locally Euclidean need only hold up to what count as infinitesimal limiting displacements.

12. It is, therefore, plausible to construe Kant's remark about setting three lines at right angles as intended to carry information about constraints on dimensionality with which the domain of actual perception may be supplemented when constructing possible locations of a sought for object.

13 The quoted passage continues as follows:

> For obviously figures could not be freely shifted and rotated in them if the curvature were not the same in all directions at all points. On the other hand, the metric properties of the manifold are completely determined by the curvature; they are therefore exactly the same in all the directions around any one point as in the directions around any other, and thus the same constructions can be effected starting from either; consequently, in the manifolds with constant curvature figures may be given any arbitrary position. (Spivak, 147)

When expanding further with discussion of *surfaces* of constant curvature Riemann points out some differences between surfaces of positive and negative curvature.

> If one regards these surfaces as possible positions for pieces of surface moving in them, as Space is for bodies, then pieces of surface can be moved in all these surfaces without stretching. The surfaces with positive curvature can always be so formed that pieces of surface can even be moved arbitrarily without bending , namely as spherical surfaces, but those with negative curvature cannot. (Spivak, 148)

14. Helmholtz, On the Facts Underlying Geometry, which originally appeared in 1868. In 1921 Paul Hertz and Mortz Schlick published a collection of Helmholtz's epistemological writings with commentaries which they provided. In 1977 Robert S. Cohen and Yehuda Elkana published a translation of this collection (Helmholtz, EW). In 1962 Morris Klein published a collection in English of lectures by Helmholtz (PSL).

15. In Riemann's day the most direct way to measure planetary distances was by triangulation with light rays by telescope observations. Where available, Radar and laser ranging now provide more direct and far more accurate distance measurements (Standish 1990). The operational definition of measurement operations as moving rigid rods into coincidences never did have much relevance to any measurements that counted as part of the practice of Astronomy.

16. Torretti (1984, 105) points out that this example of Riemann's played a central role motivating a "space-theory of matter" put forward, in 1870, by W.K. Clifford.

17. Reichenbach's book was first published, in German, in 1928. All references are to the English translation, which appeared in 1958.

18. Reichenbach may have inadvertently interchanged γ and β here, since his diagram and later discussion make it fairly clear that β is the view looking between the shells. In his diagram it is difficult to read the label on the view toward shell II. I assume that this label ought to read γ.

19. It may be worth remarking, however, that the transformation from the space corresponding to one of these projections to the space corresponding to the other would appear to reverse incongruent counterparts. It might be interesting to explore whether Reichenbach's spaces would count as orientable.

20. I have been assuming that Reichenbach's description of his space agrees with what would result from an appropriately detailed specification by systems of equations in a Riemannian manifold. I would be more confident of this assumption if it were backed up by somewhat more detailed arguments than those Reichenbach provides.

21. He needs more than a brain in a vat or Cartesian demon Hypothesis, according to which sequences perceptual experiences are recovered only as sequences of subjective episodes. The radical empiricist thesis that ultimate data are sequences of incorrigible subjective episodes might seem to allow for perceptions that are completely independent of interpretation, but I don't think Reichenbach intends, nor that his argument would be well served by, any such subjective empiricism. Kant's arguments against this thesis have been reinforced by Wittgenstein's private language argument and further arguments inspired by it.

22. The aspect of Hertz's remark I focus on is its implication that the whole figure can be surveyed from a single perspective, not what might be construed as his suggestion that the single image itself is all that is required. As we have noted above, Kant's (e.g. B 154) discussion of applications of productive imagination to represent figures we can recognize, like Helmholtz's discussion of visualization, suggests a series of successive perceptions. See DiSalle 1993 (505ff) for more on the relation between the roles of successive representations in Kant and Helmholtz.

23. This would be stronger than the senses of "*a priori*" I focused on in Harper 1986. Indeed, if an appropriate version of nativism is correct then, with appropriate limitations on scale and tolerances, propositions proved by Euclidean ostensive constructions would count as *a priori* even according to Kitcher's (1980) very restrictive interpretation.

Kant's view that ostensive constructions provide *a priori* warrant would be supported by nativism about space perception. It is not, however, obvious what, if any, sort of nativism corresponds to Kant's views on space perception (see Hatfield 1990, 104ff). Indeed, it has been argued that the nativism vs. empiricism conflict does not reflect the most interesting aspects of Kant's views on space perception (Falkenstein, 1990). On the other hand, Patricia Kitcher (1990) has been able to use nativism to illuminate quite interesting aspects of Kant's views on space perception.

24. Helmholtz appeals to our learning of our mother tongue as an example where compelling unconscious inferences are clearly supported by empirical association, rather than by any connexion given by nature (1977, 131). It is somewhat ironic that Chomsky, e.g. in his review of B.F. Skinner's *Verbal Behavior* reprinted in Fodor and Katz eds.1964, argues that our capacity to distinguish grammatical from ungrammatical strings of symbols testifies to a wired in program for a universal grammar.

In perception the so called "new look" approach emphasises the dependency of perceptual judgments on background assumptions that are open to revision (e.g. Gregory 1974). The Maar model of vision, especially as developed by Ullmann (1979), builds in commitment to Euclidean constraints relating shape and changes of images corresponding to changes of orientation to interpret visual motion.

25. Reichenbach calls this a theorem, but he provides no proof, nor any citation to a proof. He does (footnote 1) provide a formula in tensor notation. His suggestion that this counts as a theorem

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showing that undetectable forces distorting measuring rods can be specified by transformations from one metric to another is very puzzling. Einstein's discussion of apparent distortions of measuring rods corresponding to orientation with respect to spacetime curvature representing gravitation is a description of behaviour of measuring rods without distorting forces (1916 trans. in 1952, 161). The one clear case of specifying forces to make alternative assumptions about spacetime structure agree in predictions is specifications of alternative Newtonian force potentials to offset alternative assumptions about what count as inertial frames (Trautman 1965, 109-115). The physics for such a force field for relativistic gravity has not been worked out. Indeed, acceptance of the geometrization of gravitation in general relativity came after attempts to give relativistic theories of Newtonian gravitational force fields were unsuccessful (Torretti 1983, 137-143). Moreover, the predictions recovered by the alternative Newtonian force potentials are about motions of point masses (Troutman, op. cit.). What distortions such forces would produce on an extended body cannot be specified without specifying some details about the internal forces. A lead bar supported in the middle will bend more in a given gravitational field than a steel bar of the same size and shape. Such obvious differences make it incredible that Reichenbach's undetectable distorting forces can be specified by the tensor he provides.

Part of what seems to motivate Reichenbach's definition of universal force is Einstein's discussion of what he describes as the well known physical fact that uniform acceleration of free fall in a gravitational field cannot be distinguished from inertial motion (1952, 114). Where Einstein made gravity measurable as local curvature by giving up the requirement that it be interpreted as an undetectable force potential, Reichenbach defends the conventionality of metric geometry by suggesting that one could always assume undetectable distorting forces that would make coordinating definitions for an alternative metric generate the same predictions as the standard metric.

Perhaps, Reichenbach did not intend what he called his theorem to count as a detailed specification of the force. Maybe the specification of the force is as whatever is needed to distort measuring rods so that operationally defined measuring outcomes for the alternative metric will agree with those of the standard metric. This would make Reichenbach's hypothesis that such forces could be defined more like the Cartesian demon hypothesis than like any theorem in differential geometry.

Like Einstein (1952, 112-113), Reichenbach (1958, 210-218) endorses Mach's criticism of Newton's bucket experiment. Reichenbach (1958, 216-217) suggests that the coordinate independence (covariance) built into General Relativity makes rotation conventional (just a matter of choosing coordinates) and also upsets the Copernican world view by making it a matter of convention whether the earth revolves around the sun or the sun around the earth. The work by Cartan, Trautman and others on covariant spacetime formulations of Newtonian gravitation has made it clear that covariance with respect to rotating coordinate systems does not make rotation relative. In such formulations what count as the true rotations is built into the structure represented in every coordinate system (Friedman 1983, 108-114). As Einstein soon realized, the covariance of General Relativity does not make it support Mach's conjecture that all rotation is relative. (Torretti 1983, 194-202) David Malament (1985) has shown that in General Relativity rotation is fixed by the local causal structure that underwrites what Reichenbach takes to be the objectivity of topology.

26. Putnam (1975, vol. II, 161-168) argues that this sort of appeal to operationalism counts as an essentialism about meaning which, if allowed to override considerations of overall coherence, renders conventionalism trivial.

27. Newton's ideal of empirical success shares a number of features of Glymour's bootstrap confirmation, but its requirements also include hypothetico-deductive prediction and systematic dependencies which make phenomena measure parameter values rather than merely entailing the given value inferred (Harper, forthcoming). Glymour (1980, 356-364) shows that, on his notion of bootstrap confirmation, dropping the unmeasurable flat background inertial frame built into the Newtonian spacetime formulation of Newtonian gravitation (see remark on Troutman note 24) gives an empirical advantage, even though the it leads to exactly the same empirical predictions. This also happens with Newton's stronger ideal of empirical success. (DiSalle, Harper and Valluri, forthcoming)

28. Reichenbach's remark about no outside observer able to look at the *E* surface suggests that perhaps he never intended his conventionalism about geometry to extend to conventionalism about visually perceived shape.

29. Consider, for example, using a long carpenter's straight-edge level: One would need systematic forces that would bend the edge to duplicate on a flat surface gaps that would be found when moving it against different parts of surface with a ten foot hemispherical hump, even when turning it over to switch the side against the surface. One would, also, need corresponding visual images to make it look straight and rigid while undergoing all these distortions.

30. In his endorsement of Reichenbach's conventionalism Salmon (1980, 25) suggests that such conventionalism does not mean that anything goes, because once the coordinative definition is specified the correct answer can be empirically determined. To have this count as a defense against radical relativism requires strongly distinguishing between actually being able to deliver on a detailed alternative and the mere conjecture that one could, in principle, do so. As we have seen, however, it is not at all plausible that anyone could deliver on details that would make Reichenbach's proposed alternatives seriously rival the standard coordinative definitions.

31.Proper attention to the role of such phenomena in making what Reichenbach would call a coordinative definition count as viable reveals that appeal to such definitions does not lead to the conventionalism about geometry that he supposes. DiSalle (forthcoming *Erkenntnis*) discusses more general implications of this for spacetime theory as physical geometry.

32. Newton's ideal of empirical success endorses unifications according to which several phenomena give agreeing measurements of a single parameter as well as generalizations that can be appropriately backed up by measurement (Harper, forthcoming in Earman and Norton).

It can be argued that the really radical Newtonian revolution was one that installed his ideal of empirical success as a higher level standard that was later used to overthrow his theory of Universal Gravitation. General Relativity does better on Newton's standard of empirical success than Newton's theory. (Harper , op. cit.)

33. The idea of limiting common sense claims to such tolerances and scales can remove what Wilfrid Sellars (1963) characterized as a radical conflict between claims of the manifest and the scientific images. Only the failure to so limit the claims of the manifest image makes it incompatible with outcomes of scientific investigation.

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