A Watched Pot Never Boils?

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There is, in the lore of quantum mechanics, a parable that goes something like this. One evening, a long time ago, a forest elf returned to his hut, weary and hungry. He filled a pot of water, and placed it on the fire, to make stew for his supper. As he waited, he watched the pot continuously to see when it would boil. Alas, first hours . . . and then days . . . passed, and the pot never did come to a boil! The explanation is quite simple. Each observation collapsed the wavefunction of the pot into either a boiling state or non-boiling state, with relative probabilities depending on the wavefunction just prior to that observation. But, taking the limit of observations made continuously, the net probability for for ever achieving a boiling state vanishes, in the same manner as a sequence of polarizers, each with a plane of polarization rotated slightly from its predecessor, will rotate the plane of polarization of a light beam through a finite angle, with no attenuation whatever.

What is unsettling about this account is not the incident itself — for I have not the slightest doubt that it did occur, exactly as related above. Rather, it is the explanation of this incident in terms of quantum mechanics, an explanation that is inadequate in two respects. First, the explanation is merely a rough heuristic argument, e.g., employing slogans such as "collapse of the wavefunction". Preferable would be an explanation within the structure of quantum mechanics itself. Second, the explanation does not make it clear what is the difference between the elf and his pot on the one hand, and me and mine on the other (for when I place a pot of water on the fire and watch it continuously, it very definitely *does* boil).

Fix Hilbert spaces $\mathcal{H}$ (interpreted as the space of pot-states) and $\hat{\mathcal{H}}$ (interpreted as the space of instrument- (i.e., elf-) states), and consider their tensor product, $\mathcal{H} \otimes \hat{\mathcal{H}}$ (interpreted as the space of states of the combined system). Consider, on this tensor-product Hilbert space, the self-adjoint operator $H \otimes \hat{I} + P \otimes \hat{A} + I \otimes \hat{H}$, where H is some self adjoint operator on $\mathcal{H}$ (interpreted as the Hamiltonian of the pot when not under interaction), $\hat{H}$ is some self-adjoint operator on $\hat{\mathcal{H}}$ (interpreted as the Hamiltonian of the instrument when not under interaction), P is some projection operator on $\mathcal{H}$ (interpreted as the projection to "non-boiling states" of the pot, so $P^{\perp} = I - P$ is interpreted as the projection operator to "boiling states"), and $\hat{A}$ is some self-adjoint operator on $\hat{\mathcal{H}}$ (the effect of the interaction on the instrument). Thus, we interpret[1] $U_t = exp[i(H \otimes \hat{I} + P \otimes \hat{A} + I \otimes \hat{H})t]$ as the time-evolution operator for the combined pot-instrument system, under an interaction we interpret as "continuous observation by the instrument as to whether the pot is boiling". We demand, finally, that $\hat{H}$ and $\hat{A}$ commute with each other (which we interpret as the demand that the instrument's records not be altered in time via evolution of the instrument). Consider now the element $\psi \otimes \hat{\mu} \in H \otimes \hat{H}$, where ψ is some unit vector in $\mathcal{H}$ (interpreted as the initial pot-state) satisfying $P\psi = \psi$ (interpreted as meaning that ψ is a "non-boiling" state), and $\hat{\mu}$ is some unit vector in $\hat{\mathcal{H}}$ (interpreted as the initial instrument state).

The key theorem, whose proof we defer to the end, is the following:

Theorem. Under the conditions of the previous paragraph, we have, for every $t \geq 0$, the bound

$$\|(P^{\perp} \otimes \hat{I})U_t(\psi \otimes \hat{\mu})\|^2 \leq [4\ |H|^2\ t\ e^{4|H|t}] \int_0^t d\tau\ | < \hat{\mu}|e^{i\hat{A}\tau}|\hat{\mu} > |. \quad (1)$$

The physical meaning of this theorem is the following.

The left side of Eqn. (1) is the result of taking the initial state, $\psi \otimes \hat{\mu}$, evolving for time t, projecting into the "boiling subspace", and taking the squared norm. This side, in short, may be interpreted as "the probability that the pot is boiling, as a function of time t" (understood, in the usual way, as the result of an observation made on an ensemble).

The factor in square brackets on the right involves only the elapsed time t and the norm of the pot-Hamiltonian, $|H|$. Note that this factor can become very large. For example, for a typical pot (reasonably approximated by a

[1] It is necessary in what follows to demand that the operator H be bounded. However, the operators $\hat{H}$ and $\hat{A}$ may be unbounded, provided only that all linear combinations of them are self-adjoint.

bounded Hamiltonian), and an elapsed time of ten minutes, this factor is about $10^{10^{40}}$/sec.

The integral on the right in Eqn. (1) involves only the elapsed time t, the interaction operator $\hat{A}$ on the instrument Hilbert space, and the initial instrument state $\hat{\mu}$. This integral will be small provided that evolution under the operator $\hat{A}$ as "Hamiltonian" sends the initial state $\hat{\mu}$ to one that quickly becomes and remains nearly orthogonal to $\hat{\mu}$. This, in turn, tends to happen provided that "the operator $\hat{A}$ is sufficiently large and diffuse in its effects on the Hilbert space $\hat{\mathcal{H}}$, while the initial state $\hat{\mu}$ is in general position with respect to $\hat{A}$." Here is an exmple. Let $\hat{\mathcal{H}} = L^2(R)$, let $\hat{A} = iv\ d/dx$ where v is some constant, and let $\hat{\mu}$ be a wave function that vanishes outside some interval of length a. Then, for all $t > 0$, we have $\int_0^t d\tau\ | < \hat{\mu}|e^{i\hat{A}\tau}|\hat{\mu} > | \leq a/v$. This can be made small by choosing the width, a, of the initial wave packet small and/or choosing the size, v, of $\hat{A}$ large.

The theorem above, then, provides an explanation, within quantum mechanics, for the failure of the elf's pot to boil. But note what an extrordinary elf he must have been! He was apparently able to adjust his initial state $\hat{\mu}$ and/or his interaction operator $\hat{A}$ so finely as to achieve an integral on the right in Eqn. (1) down from one sec by about 10^{40} orders of magnitude.

Does there exist a bound significantly better than that of the theorem? I'm not sure. It is not difficult to modify the proof (below) so as to replace the factor "$|H|^2$" on the right by "$|[H,P]|^2$", but that does not appear to reduce significantly the numerical size of this factor. [Replacing all $|H|$'s by $|[H,P]|$'s presumbly would, but I doubt that this can be achieved.] Is there some analogous theorem – having similar physical consequences – that works without the assumption that H be bounded; or without the assumption that $\hat{A}$ and $\hat{H}$ commute? I'm not sure. My guess is that there may indeed be analogous bounds in these cases, but that, if they exist at all, they will be far more complicated than that of Eqn. (1).

Proof of the theorem.

The key formula is the following

$$U_t = P\ e^{iHPt} \otimes e^{i(\hat{A}+\hat{H})t} + P^{\perp} e^{iHP^{\perp}t} \otimes e^{i\hat{H}t} \tag{2}$$

$$+ \int_0^t d\tau\ C(t,\tau) \otimes e^{i\hat{A}\tau} e^{i\hat{H}t}. \tag{3}$$

where the operator $C(t,\tau)$ on $\mathcal{H}$ is given by the sum

$$C(t,\tau) = \sum i^{m+n+1}\ P^? H P^? H \cdots H P^? \tau^n (t-\tau)^m / n! m!. \tag{4}$$

Here, each of the symbols "$P^?$" on the right stands for either P or $P^\perp$, and $n+1$ is the number of P's that appear in that term, $m+1$ the number or $P^\perp$'s. The sum is over all possible arrangements provided only that $m \geq 0, n \geq 0$, i.e., that there appear at least one "P" and one "$P^\perp$" in each term. Eqn. (2) can be established in a number of ways, all of which are relatively tedious: i) Note that this equation holds for $t = 0$, and then, taking d/dt of both sides, show that the time derivative of this equation holds at t if the equation itself holds at t. ii) Expand the left side of Eqn. (2) in power series, and collect the terms according to the power or H they contain. iii) Manipulate the standard formula, involving the time-ordered products, for the effect of an interaction (taken in this case as $H \otimes \hat{I}$) on the time-evolution of a system. In this regard, the following little formula is useful:

$$e^{i(P\otimes\hat{A}+I\otimes\hat{H})t} = P \otimes e^{i(\hat{A}+\hat{H})t} + P^\perp \otimes e^{i\hat{H}t}. \tag{5}$$

Applying, to each side of Eqn. (2), $P^\perp \otimes \hat{I}$ from the left and $\psi \otimes \hat{\mu}$ from the right, the first two terms on the right side of that equation drop out. Taking the norm of what remains, we obtain

$$\|(P^\perp \otimes \hat{I}) U_t(\psi \otimes \hat{\mu})\|^2 \leq <\psi|C(t,\tau)P^\perp C(t,\tau)|\psi> \int_0^t d\tau \int_0^t d\tau'\ |<\hat{\mu}|e^{i\hat{A}(\tau-\tau')}|\hat{\mu}>|. \tag{6}$$

The result is now a consequence of the following bound:

$$|P^\perp C(t,\tau)P| \leq \sum_{m,n=0}^{\infty} |H|^{m+n+1} \tau^n (t-\tau)^m (m+n)!/(m!n!)^2 \tag{7}$$

$$\leq \sum_{m,n=0}^{\infty} |H|^{m+n+1} \tau^n (t-\tau)^m\ 2^{m+n}/m!n! \tag{8}$$

$$= |H| e^{2|H|t}. \tag{9}$$

In the first step, we used the fact that the number of terms on the right in Eqn. (4) with given m,n-values is $(m+n)!/m!n!$; in the second, the fact that $(m+n)!/m!n! \leq 2^{m+n}$.

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