Path Integrals

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1 Introduction

2 General Issues

Consider a simple physical system. That is, let there be given a set X, the configuration space, and a real, nonnegative measure $\mathcal{N}, \nu$ on X. (Here, $\mathcal{N}$ is the collection of measurable subsets of X with finite measure, and $\mathcal{N} \xrightarrow{\nu} R^+$ gives the value of the measure). Then $L^2(X, \nu)$, the collection of square-integrable, complex-valued functions on X, is a Hilbert space. Further, let there be given a family, U_t, of bounded operators on this Hilbert space, labeled by the number $t > 0$. We require that this family satisfy[1] the following property: For any $t, t' > 0$, we have $U_t \circ U_{t'} = U_{t+t'}$.

Fix a positive number T (the "total elapsed time"). It is convenient to introduce a special notation for certain matrix elements of certain operators. Let s be any positive integer, $\mathbf{t} = (t_0, \cdots, t_s)$ any $(s+1)$-tuple of real numbers with $0 = t_0 < t_1 < \cdots < t_s = T$, and $\mathbf{A} = (A_0, \cdots, A_s)$ any $(s+1)$-tuple of measurable subsets of X with A_0 and A_s each having finite measure. Thus, $\mathbf{t}$ is a partition of the interval $[0, T]$, and $\mathbf{A}$ assigns a measurable set to each point of the partition. Given these things, set

$$\sigma(\mathbf{t}, \mathbf{A}) = < \chi_{A_0} | U_{t_1 - t_0} Q_{A_1} U_{t_2 - t_1} Q_{A_2} \cdots U_{t_s - t_{s-1}} | \chi_{A_s} >, \qquad (1)$$

where χ_A denotes the characteristic function of the set $A \subset X$, and Q_A denotes the configuration operator of A (i.e., the operator that multiplies

[1] In most examples, these U_t will also be unitary, but this property will not be needed in what follows.

a wave function by χ_A). Note that the conditions we have imposed on $\mathbf{t}, \mathbf{A}$ are precisely what are needed in order that the right side of 1 make sense. This σ has the following properties. i) Let $\mathbf{t}', \mathbf{A}'$ result from $\mathbf{t}$ and $\mathbf{A} = (A_0, \cdots, X, \cdots, A_s)$ by omitting one entry of "X" from $\mathbf{A}$, and also the corresponding entry from $\mathbf{t}$. Then $\sigma(\mathbf{t}', \mathbf{A}') = \sigma(\mathbf{t}, \mathbf{A})$. [This follows from the fact that Q_X =identity.] ii) Fix partition $\mathbf{t}$, and let $\mathbf{A}, \mathbf{A}', \mathbf{A}''$ be identical except for one slot, in which the entries of $\mathbf{A}$ and $\mathbf{A}'$ are disjoint and that of $\mathbf{A}''$ is their union. Then $\sigma(\mathbf{t}, \mathbf{A}'') = \sigma(\mathbf{t}, \mathbf{A}) + \sigma(\mathbf{t}, \mathbf{A}')$. [This follows from the fact that, $Q_A + Q_B = Q_{A \cup B}$ whenever $A \cap B = \emptyset$.] iii) We have the bound $|\sigma(\mathbf{t}, \mathbf{A})| \leq [\nu(A_0)\nu(A_s)]^{1/2} |U_{t_1 - t_0}| \cdots |U_{t_s - t_{s-1}}|$.

Here are three examples.

For the first, let all the operators U_t be the identity operator. Then $\sigma(\mathbf{t}, \mathbf{A}) = \nu(A_0 \cap \cdots \cap A_s)$.

For the second example, let $X = R^n$ and ν ordinary Lebesque measure, and let the operators U_t be the evolution operators for the heat equation. Then, using the Green's function for the heat equation, we have

$$\sigma(\mathbf{t}, \mathbf{A}) = [(2\pi\kappa)^s (t_1 - t_0) \cdots (t_s - t_{s-1})]^{-n/2} \tag{2}$$

$$\int_{(x_0, \cdots, x_s) \in A_0 \times \cdots \times A_s} exp[-\{(x_1 - x_0)^2/(t_1 - t_0) + \cdots + (x_s - x_{s-1})^2/(t_s - t_{s-1}\})/2\kappa] \tag{3}$$

$$d\nu_{x_0} \cdots d\nu_{x_s}, \tag{4}$$

where $\kappa > 0$ is the thermal conductivity. Note that the integral on the right, taken over a certain subset of $X \times \cdots \times X$ ($s + 1$ times), always converges.

For the third example, again let $X = R^n$ and ν Lebesque measure. But now let the operators U_t be the evolution operators for the Schroodinger equation for a free, mass-m, particle. Then, using the Green's function for the Schroodinger equation, we have

$$\sigma(\mathbf{t}, \mathbf{A}) = [(2i\pi\hbar/m)^s (t_1 - t_0) \cdots (t_s - t_{s-1})]^{-n/2} \tag{5}$$

$$\int_{(x_0, \cdots, x_s) \in A_0 \times \cdots \times A_s} exp[im\{(x_1 - x_0)^2/(t_1 - t_0) + \cdots + (x_s - x_{s-1})^2/(t_s - t_{s-1})\}/2\hbar] \tag{6}$$

$$d\nu_{x_0} \cdots d\nu_{x_s}. \tag{7}$$

Now, however, the right side cannot be regarded as an integral over a certain subset of $X \times \cdots \times X$ ($s+1$ times), for this integral fails to converge in general.

[To see this, recall that an integral converges if and only if it converges with the integrand replaced by its absolute value. But here this absolute value is 1.] Instead, the right side must be interpreted as an iterated integral, i.e., on that is to be carried out over the individual variables, $x_0, \cdots, x_s$, one at a time and in a certain specified order. The correct order in this case is: first x_0 and x_s, then x_1 and x_{s-1}, etc. Done in this way, convergence at each step is guaranteed, and so the right side makes sense. It is this "right side", so defined, that yields $\sigma(\mathbf{t}, \mathbf{A})$.

We now turn to spaces of paths. Denote by Ω the collection of all paths, $[0,T] \xrightarrow{\gamma} X$, in X, parameterized by time t in the interval $[0,T]$. Note that we are including in Ω *all* paths, even, e.g., those that are discontinuous. For any $\mathbf{t}, \mathbf{A}$ as above, write $\Omega_{\mathbf{t},\mathbf{A}}$ for the collection of all such paths γ with $\gamma(t_0) \in A_0, \cdots, \gamma(t_s) \in A_s$. Note that membership in $\Omega_{\mathbf{t},\mathbf{A}}$ restricts the path only at the times determined by the partition $\mathbf{t}$. We call these subsets $\Omega_{\mathbf{t},\mathbf{A}}$ of Ω *regular* subsets. Note that any finite intersection of regular subsets is another, and that any difference or finite union of regular subsets can be written as a finite union of disjoint regular subsets. Now denote by $\mathcal{M}$ the collection of all finite unions of regular subsets of Ω. Let $\mathcal{M} \xrightarrow{\mu} C$ assign, to each regular subset $\Omega_{\mathbf{t},\mathbf{A}}$, the number $\sigma(\mathbf{t}, \mathbf{A})$; and to each finite union of disjoint regular subsets the sum of these corresponding numbers (noting that these rules define μ, unambiguously, on all sets in $\mathcal{M}$.) The resulting $\mathcal{M}, \mu$ is a premeasure space (in the sense of the Appendix, i.e., essentially, μ is finitely additive).

The key question is this: When does this premeasure, $\mathcal{M}, \mu$, on Ω give rise to a full measure on Ω? This question is answered in Appendix B: A necessary and sufficient condition is given by Theorem 2. In order to adapt that theorem to the present context, it is convenient to introduce the following notation. Fix $\mathbf{t}, \mathbf{A}$, and then set

$$|\sigma|(\mathbf{t}, \mathbf{A}) = lub \sum |\sigma(\mathbf{t}', \mathbf{A}')|. \tag{8}$$

Here, the sum is over some finite collection of $\mathbf{t}', \mathbf{A}'$ such that the corresponding $\Omega_{\mathbf{t}',\mathbf{A}'}$ are disjoint subsets of $\Omega_{\mathbf{t},\mathbf{A}}$. The least upper bound is then over all such finite collections. The value of the right side of 8 can, of course, be specified directly, without reference to spaces of curves: First fix partition $\mathbf{t}' \supset \mathbf{t}$, with s' entries, and regard $\mathbf{A}$ as defining a subset or $X \times \cdots \times X$ ($s'+1$ times). Then take the sum above over a finite collection of corresponding $\mathbf{A}'$

that (regarded as subsets of $X \times \cdots \times X$ ($s'+1$ times)) are disjoint subsets of $\mathbf{A}$. Now the least upper bound is over such finite collections (for fixed $\mathbf{t}' \supset \mathbf{t}$), and also over all such partitions $\mathbf{t}'$. Thus, the $|\sigma|(\mathbf{t}, \mathbf{A})$ that results from Eqn. (8) is a nonnegative number (where the value "infinity" is also allowed, in case the least upper bound fails to exist).

We now have the following

Theorem 1. The above premeasure $\mathcal{M}, \mu$ on Ω gives rise to a measure on Ω if and only if the following conditions are satisfied. Given any $A, B \in \mathcal{N}$,

1. $|\sigma|(\mathbf{t}, \mathbf{A})$ is finite, where $s = 1$, $\mathbf{t} = (0, T)$ and $\mathbf{A} = (A, B)$; and

2. given, in addition, any measurable $C_1 \supset C_2 \supset \cdots \in \mathcal{N}$ with $\cap C_i = \emptyset$, and any number $0 \leq t_1 \leq T$, we have $|\sigma|(\mathbf{t}', \mathbf{A_i}) \to 0$ as $i \to \infty$, where[2] $s = 2$, $\mathbf{t}' = (0, t_1, T)$, and $\mathbf{A_i} = (A, C_i, B)$.

Proof: First note that $|\sigma|(\mathbf{t}, \mathbf{A}) = |\mu|(\Omega_{\mathbf{t},\mathbf{A}})$, where on the right is the variation of the premeasure (as defined in Appendix B). Then condition 1 of the Theorem is precisely the statement that the premeasure $\mathcal{M}, \mu$ have bounded variation. Furthermore, condition 2 of the Theorem is precisely the statement that this premeasure have the nesting property. [This is seen as follows. Consider any nested sequence, α_i, of sets in $\mathcal{M}$, with empty intersection in Ω. Then, since Ω consists of *all* paths in X, there must be some number $0 \leq t_1 \leq T$ such that $\cap C_i = \emptyset$, where $C_i \subset X$ is the subset resulting from evaluating the paths in α_i at time t_1. We further have $C_1 \supset C_2 \supset \cdots$.] The theorem itself is now merely a restatement of Theorem 2 of Appendix B.

Note that the first condition of Theorem 1 is precisely the requirement that *all* the $|\sigma|(\mathbf{t}, \mathbf{A})$ are finite. The second condition may be understood as follows. First note that, given any A, B, t_1 and C_i as in this condition, it follows automatically that $|\sigma(\mathbf{t}', \mathbf{A_i})| \to 0$ as $i \to \infty$. But condition 2 itself requires something much stronger, namely that $|\sigma|(\mathbf{t}', \mathbf{A_i}) \to 0$.

Let us now turn, again, to our three earlier examples. For the first (with U_t = identity), we have $|\sigma|(\mathbf{t}, \mathbf{A}) = \sigma(\mathbf{t}, \mathbf{A}) = \nu(A_0 \cap \cdots \cap A_s)$. Then condition 1 of the theorem is immediate from this formula, and condition 2 follows directly from the fact that ν is a measure on X. Hence, by the Theorem, there is a measure on Ω in this example. For the second (heat

[2]The two limiting values of t_1 are to be treated as follows. If $t_1 = 0$, then instead set $s = 1$, $\mathbf{t}' = (0, T)$, and $\mathbf{A_i} = (A \cap C_i, B)$, and similarly if $t_1 = T$.

equation), we again have $|\sigma|(\mathbf{t}, \mathbf{A}) = \sigma(\mathbf{t}, \mathbf{A})$, given by the right side of Eqn. (2). Again, the two conditions of the theorem follow directly, and so, again, we obtain a measure on Ω. This is Wiener measure. Finally, for the third example (Schroodinger equation), we have $|\sigma|(\mathbf{t}, \mathbf{A}) = \infty$ in all cases in which $\nu(A_0)$ and $\nu(A_s)$ are nonzero. This follows from the failure of the integral, regarded as over a subset of $X \times \cdots \times X$, on the right in Eqn. (5) to converge. So, in this case neither condition 1 nor condition 2 is satisfied, and we do not have a corresponding measure on the space Ω of paths. I do not know of an example in which condition 1 of Theorem 1 holds, while condition 2 fails.

Note that the conditions of the Theorem refer *only* to properties of the matrix elements themselves, as reflected in the function σ, and not to any more abstract properties of path spaces. Thus, given a system — described by a measure space $X, \mathcal{N}, \nu$ together with a family U_t of operators on $L^2(X, \nu)$ — it is in some sense an elementary calculation to check the conditions of the theorem, and thus to decide whether or not that system admits a corresponding measure on path space. But, unfortunately, things are not normally this simple in practice. The problem is that it can be difficult in practice to evaluate the function $|\sigma|$ given in Eqn. (8), because of the least upper bound on the right. The following simplification is sometimes useful. Fix $\mathbf{t}, \mathbf{A}$. Now define the number $\widehat{|\sigma|}(\mathbf{t}, \mathbf{A})$ by the right side of Eqn. (8), but with there imposed the further restriction that $\mathbf{t}' = \mathbf{t}$ in the least upper bound on the right. That is, we now allow only the partition $\mathbf{t}$ itself of $[0, T]$, but no longer refinements of it. By virtue of this restriction, the evaluation of $\widehat{|\sigma|}$ considerably simpler than of $|\sigma|$. Clearly, we have $\widehat{|\sigma|}(\mathbf{t}, \mathbf{A}) \leq |\sigma|(\mathbf{t}, \mathbf{A})$. Hence, the conditions of the theorem, imposed on $\widehat{|\sigma|}$ rather than on $|\sigma|$, are now merely necessary — but need no longer be sufficient — for there to exist an extension of the premeasure to a measure on path-space Ω. Evaluation of $\widehat{|\sigma|}$ is often sufficient to conclude that there is no appropriate measure on Ω.

We may interpret this simplification in the following way. Fix s and $\mathbf{t}$. Then with each corresponding $\mathbf{A}$ we may associate a subset, $A_0 \times \cdots \times A_s$, of $X \times \cdots \times X$ ($s+1$ times). Assign to each such subset the number $\sigma(\mathbf{t}, \mathbf{A})$, thus defining a premeasure on $X \times \cdots \times X$ ($s + 1$ times). Then, still fixing $\mathbf{t}$, the conditions of the theorem are the necessary and sufficient conditions that this premeasure extend to a measure on $X \times \cdots \times X$ (($s + 1$) times). Replacing the partition $\mathbf{t}$ by a finer partition results in a new premeasure on Ω that extends the original premeasure. The issue of whether there is a measure on all of Ω, as given in Theorem 1, is then whether the resulting family of premeasures on Ω (ordered by extension) have a suitable limit. In the case of a free particle, to take one example, we do obtain a measure in the case $s = 1$ (i.e., on $X \times X$), but not for any larger s-value. It is interesting to ask whether there are any examples of systems for which even the matrix elements of U_T between initial and final states arise from no measure on $X \times X$. It turns out that there is such an example, that of a bead sliding about a circular wire. This example is discussed in Appendix A.

Finally, we discuss briefly the issue of whether restrictions may be imposed on the paths, still retaining a measure in the sense of the Theorem. The only place we used that Ω consists of all paths was in translating the second condition to the nesting property. Thus, we have a theorem similar to that of the Theorem, but with i) the path-space Ω replaced by any fixed collection of paths of X; and ii) condition 2 replaced by the statement that the premeasure $\mathcal{M}, \mu$ on Ω satisfy the nesting property of Appendix B. Thus, in the first example above, it suffices to take for Ω the space consisting only of the constant paths. As a second example, it is known (ref) that, in the example of Weiner measure for the heat equation, Ω can be replaced by all paths satisfying a Lipschitz condition of order 1/2. (This merely reflects the fact that initial data for the heat equation "spreads out" at a rate proportional to $t^{1/2}$, for small t.) As a second example, consider the case in which all the U_t are just the identity operator. In that case, we may replace Ω by a very small collection of paths: those of the form $\gamma(t) = \hat{x}$, i.e., the constant paths. The measure on this collection of paths, guaranteed by the theorem, is then of course precisely the original measure ν on X.

3 A Class of Operators

Fix, again, a set X and a nonnegative measure, $\mathcal{N}, \nu$, on X, and consider, again, the Hilbert space $L^2(X, \nu)$. We now introduce a particularly useful class of operators on this Hilbert space.

Denote by $\mathcal{P}$ the collection of all complex-valued, measurable functions, $X \times X \xrightarrow{p} C$, such that, for some positive number κ_p, we have

$$\int_{x \in X} |p(x, y)| \; d\nu_x \leq \kappa_p, \quad \int_{x \in X} |p(y, x)| \; d\nu_x \leq \kappa_p \tag{9}$$

for almost all $y \in X$ (so, in particular, these two integrals must converge). Let us agree to choose for κ_p the *smallest* number that suffices in Eqn. (**??**). We now claim that each element p of $\mathcal{P}$ defines an operator, which we write O_p, on $L^2(X,\nu)$, given by the following formula:

$$< \phi|O_p|\psi >= \int_{x,y\in X} \overline{\phi}(x)p(x,y)\psi(y)\ d\nu_x d\nu_y. \tag{10}$$

for all $\phi, \psi \in L^2(X,\nu)$. That is, we claim that the integral on the right always converges, and yields the matrix elements of some operator O_p. To see this, first note that

$$\begin{aligned} |\int_{x,y\in X} \overline{\phi}(x)p(x,y)\psi(y)\ d\nu_x d\nu_y| &\leq (1/2)\int_{x,y\in X} |p(x,y)|(b|\phi(x)|^2 + |\psi(y)|^2/b)\ d\nu_x d\nu_y \\ &\leq (\kappa_p/2)(b||\phi||^2 + ||\psi||^2/b). \end{aligned} \tag{11}$$

In the first step, we used the Schwarz inequality (where b is any positive number). In the second, we used the Fubini theorem — integrating the y-variable first in the first term and the x-variable first in the second — and Eqn. (**??**). Finally, choosing $b = ||\psi||/||\phi||$ on the right in Eqn. (11), we conclude that the left side of (11) is bounded by $\kappa_p\ ||\psi||\ ||\phi||$. But this implies, in turn, that there exists an operator O_p — clearly unique — satisfying (10). Note[3] that we have $O_p = O_{p'}$ if and only if $p = p'$ almost everywhere on $X \times X$.

We now derive some properties of this class[4] of operators.

1. The operator O_p is bounded, and, in fact, $|O_p| \leq \kappa_p$. This is immediate from the proof, above, of convergence of the right side of Eqn. (9).

2. These operators form a vector space. Indeed, for any $p, p' \in \mathcal{P}$ and $c \in C$, we have that $p + cp' \in \mathcal{P}$, and $O_{p+cp'} = O_p + cO_{p'}$.

[3]So, we could as well have defined $\mathcal{P}$ using equivalence classes of functions p, under the equivalence relation of equality almost everywhere, noting that κ_p depends only on equivalence class.

[4]There exists in fact a slightly more general class of operators having similar properties. Consider a measure p on $X \times X$ for which there exist X-dependent measures α_x, β_x (for all $x \in X$) on X such that: i) For some positive number a, we have $|\alpha_x|(X) \leq a$ and $|\beta_x|(X) \leq a$, for all $x \in X$; and ii) for any measurable $A, B \subset X$, we have $\int_{x\in A} \alpha_x(B)\ d\nu_x = \int_{y\in B} \beta_y(A)\ d\nu_y = p(A \times B)$. Then this measure p gives rise to an operator in a manner similar to (9). The version above will be recognized as the special case in which the measure is given by $p(x,y)\ d\nu_x d\nu_y$. We have not adopted this generalization because it adds more in complication than in content.

3. These operators are closed under taking adjoints. Indeed, for any $p \in \mathcal{P}$, we have, setting $p^\dagger(x,y) = \overline{p(y,x)}$, that $p^\dagger \in \mathcal{P}$, and $(O_p)^\dagger = O_{p^\dagger}$.

4. These operators are closed under composition. We first claim that, for $p, p' \in \mathcal{P}$, there is an element, $p * p' \in \mathcal{P}$, given by $(p*p')(x,y) = \int p(x,z)p'(z,y)\ d\nu_z$. To see this, first note that $\int |p(x,z)|\ |p'(z,y)|\ d\nu_y d\nu_z \leq \kappa_p \kappa_{p'}$ (a consequence of Fubini's theorem and (**??**)). But this implies, again by Fubini's theorem, that the $p * p'$ given above is well-defined for almost all x, y, and is in $\mathcal{P}$. In fact, we have $\kappa_{p*p'} \leq \kappa_p \kappa_{p'}$. Finally, we note that the convolution operation on $\mathcal{P}$ thus defined reflects composition of operators: $O_p \circ O_{p'} = O_{p*p'}$.

5. These operators are closed under taking limits, in the following sense. Let $p_1, p_2, \cdots \in \mathcal{P}$ be Cauchy in the norm $|\ |$ (i.e., for every $\epsilon > 0$ there exists a number N such that $|p_i - p_j| \leq \epsilon$ whenever $i, j \geq N$). Then, we claim, there exists $p \in \mathcal{P}$ such that $|p_i - p| \to 0$ as $i \to \infty$. To see this, first note, from the Cauchy property, that there exists a function $\hat{p}$ on $X \times X$ such that $p_i(x,y) \to \hat{p}(x,y)$ in $L^1(X)$ (variable x), uniformly in $y \in X$; and, similarly, a function $\check{p}$ such that $p_i(x,y) \to \check{p}(x,y)$ in $L^1(X)$ (variable y), uniformly in $x \in X$. But this implies, by the Fubini theorem, that, for any $A, B \in \mathcal{N}$, $\int_{(x,y)\in A\times B} p_i(x,y)\ d\nu_x d\nu_y \to \int_{(x,y)\in A\times B} \hat{p}(x,y)\ d\nu_x d\nu_y$, as $i \to \infty$, and similarly for $\check{p}$. Hence, $\hat{p} = \check{p}$ almost everywhere in $X \times X$. Let $p \in \mathcal{P}$ be this common value almost everywhere. Then we have $|O_{p_i} - O_p| \to 0$, i.e., that the O_{p_i} converge in norm to O_p.

6. These operators are dense in the space of all bounded operators, in the following (what is usually called the "strong") sense. Let A be any bounded operator on $L^2(X,\nu)$. Then there exists a sequence, $p_1, p_2, \cdots \in \mathcal{P}$, such that, for every $\psi \in L^2(X,\nu)$, we have $||(A - O_{p_i})\psi|| \to 0$ as $i \to \infty$. This assertion in turn is a consequence of the following three facts. First, the collection of all $\psi \in L^2(X,\nu)$ that are both bounded and in $L^1(X,\nu)$ is dense in $L^2(X,\nu)$. (To see this, note that the finite linear combinations of characteristic functions on sets in $\mathcal{N}$ are bounded and dense in $L^2(X,\nu)$.) Second, every function of the form $p(x,y) = \sum_{i,j=1}^n a_{ij}\overline{\psi_i}(x)\psi_j(y)$, with the ψ_i both bounded and in $L^1(X,\nu)$, is in $\mathcal{P}$. Third, the set of operators arising from functions p of this form is dense, in the sense given above, in the space of all bounded operators on $L^2(X,\nu)$.

Note that the set of operators of the form O_p is closed (property 5) in a different topology from that in which this set is dense (property 6). Thus, there are in general many bounded operators (such as the identity) not among the O_p.

Here is an example of an application of these properties. Let $X \times X \xrightarrow{p} C$ be any function in $\mathcal{P}$ satisfying $\overline{p(x,y)} = p(y,x)$. Then the corresponding operator O_p is self-adjoint. Now consider, for each $t > 0$, the operator

$$\hat{U}_t = exp(itO_p) = I + itO_p/1! + (it)^2 O_p \circ O_p/2! + \cdots = I + O_w, \qquad (12)$$

where $w \in \mathcal{P}$ is the function given by $w = itp/1! + (it)^2 p * p/2! + \cdots$. These sums converge, by properties 1, 2, 4, and 5 above. The $\hat{U}_t$ so defined is a family of unitary operators on $L^2(X,\nu)$, and they automatically satisfy the semigroup property: $\hat{U}_t \circ \hat{U}_{t'} = \hat{U}_{t+t'}$.

We now wish to work out the matrix elements, $\sigma(\mathbf{t}, \mathbf{A})$, of Eqn. (1), for the family $\hat{U}_t$ of unitary operators given above. This is straightforward: Substitute $I + O_w$ from Eqn. (9) for each $\hat{U}_t$, and expand. Each operator I, sandwiched between configuration operators, drops out via $Q_{A_i} I Q_{A_{i+1}} = Q_{A_i \cap A_{i+1}}$. We are thus left with integrals of certain functions w over certain subsets of $X \times \cdots \times X$. But an integral of w is less than or equal to that integral of $|w|$, which is less than or equal to that integral of $t|p|/1! + t^2 |p| * |p|/2! + \cdots$. We conclude that

$$|\sigma(\mathbf{t}, \mathbf{A})| \leq < \chi_{A_0} | e^{(t_1 - t_0) O_{|p|}} Q_{A_1} \cdots Q_{A_{s-1}} e^{(t_s - t_{s-1}) O_{|p|}} | \chi_{A_s} > . \qquad (13)$$

From this and the definition of $|\sigma|$, we have

$$|\sigma|(\mathbf{t}, \mathbf{A}) \leq < \chi_{A_0} | e^{(t_1 - t_0) O_{|p|}} Q_{A_1} \cdots Q_{A_{s-1}} e^{(t_s - t_{s-1}) O_{|p|}} | \chi_{A_s} > \leq < \chi_{A_0} | e^{T O_{|p|}} | \chi_{A_s} > . \qquad (14)$$

It follows immediately from Eqn. (14) that the hypothesis of Theorem 1 is satisfied for the family U_t of operators given by Eqn. (1).

Thus, we have shown: For any family $\hat{U}_t$ of operators given by Eqn. (12) with $p \in \mathcal{P}$ satisfying $\overline{p(x,y)} = p(y,x)$, the premeasure σ on the space Ω of paths extends automatically to a full (complex-valued) measure on Ω.

We next claim that, given a physical system with configuration space X, and quantum space of states $L^2(X,\nu)$; with evolution operators U_t, we can always find a $p \in \mathcal{P}$ such that the family $\hat{U}_t$ of operators given by Eqn. (11) is close to the U_t, in the sense of property 6 above. For example, consider the case in which $X = R^3$, and our system is a particle in a (say, bounded) potential V. A suitable function p in this case might be that given as follows. Set $f(x) = (2\pi\rho)^{-1/2} exp(-x^2/2\rho)$, for $\rho > 0$. Now set

$$p(x,y) = -\hbar^2/2m \; f''(x-y) + (V(x) + V(y)) f(x-y)/2. \qquad (15)$$

Then, as $\rho \to 0$, we have, uniformly in t, that $\hat{U}_t \to U_t$, in the sense of property 6 above.

The crucial question, then, is whether these families $\hat{U}_t$ of operators for which a measure over path-space exists "approximate" the actual evolution family U_t in a sense sufficient for physical applications. Blah, blah, blah

Appendix A - Bead on a Wire

In this appendix, we give an example of a simple mechanical system whose evolution cannot be described by any path integral — even in the weak sense described in Sect. 1. That is, this system will be such that there is no measure $\mathcal{M}, \mu$ on $X \times X$ that reproduces the transition amplitudes.

Let X be the circle of radius R (labeled by coordinate $\theta \in [0, 2\pi]$). Let the Hamiltonian be that of a free particle of mass m, i.e., $H = (1/2m)(\hbar/iR)^2\partial^2/\partial\theta^2$. Thus, this system consists, physically, of a bead free to slide about a circular wire. Schroodinger's equation, on wave function $\psi(\theta, t)$, is $-\omega/2\ \partial^2\psi/\partial\theta^2 = i\partial\psi/\partial t$, where we have set $\omega = \hbar/mR^2$. Fix a time $T > 0$, and denote by U_T the corresponding unitary time-evolution operator.

We now claim: There exists (except possibly for a certain countable collection of T-values, discussed below) no measure μ on $X \times X$ (i.e., on the torus) such that, for every measurable $A, B \subset X$, we have $\mu(A \times B) =< \chi_A|U_T|\chi_B >$.

A somewhat weaker version of this claim — that there can be no bounded Green's function for U_T — is actually quite easy to prove. Indeed, suppose, for contradiction, that there were one, $G(\theta, \theta')$, so we would have $< \phi|U_T|\psi >= \int_{X\times X} \overline{\phi}(\theta)G(\theta, \theta')\psi(\theta')\ d\nu_{\theta'}d\nu_\theta$. Choosing $\psi \neq 0$ supported on measurable $A \subset X$, we have, using the Schwartz inequality, $| < \phi|U_T|\psi > | \leq ||\phi||\ ||\psi||\ [\int_{X\times A} |G(\theta, \theta')|^2\ d\nu_\theta d\nu_{\theta'}]^{1/2}$. Now set $\phi = U_T\psi$ in this inequality, so, by unitarity, the left side is $||\psi||^2$, the first factor on the right $||\psi||$. Letting $\nu(A) \to 0$ results in a contradiction[5] [6].

[5] Actually, we have shown more. Consider any simple mechanical system having measurable $A_1 \supset A_2 \supset \cdots \subset X$, each of positive measure, with $\cap A_i = \emptyset$. Then this system possesses no Green's function, $G(x, x')$, lying in $L^2(X \times X)$.

[6] There is another argument that again suggests that there will be no Green's function for this problem. If there is to be a Green's function, then one would expect that it be given by $G(\theta_1, \theta_2) = [2\pi i\omega T]^{-1/2} \sum e^{i(\theta_2-\theta_1+2\pi n)^2/2\omega T}$, where the sum is over all integers

We now proceed to demonstrate nonexistence of *any* measure that reproduces the transition amplitudes as above. To see this, we first note that our system has two symmetries. For $\psi(\theta, t)$ any solution of Schroodinger's equation, so are i) $\psi(\theta+\theta_o, t)$, for every $\theta_o \in [0, 2\pi]$; and ii) $\exp(in\theta - in^2\omega t/2)\psi(\theta - n\omega t, t)$, for n any integer (a requirement necessary for single-valuedness of ψ). The first is the simple rotational symmetry of the circular wire as a whole. The second corresponds to a rotation that "increases linearly in time", accompanied by a θ-dependent phase shift. This symmetry is an adaptation, to circular geometry, of the (Galilean) "boost symmetry" for an ordinary free particle. It is an immediate consequence of these symmetries that the unitary time-evolution operator, U_T, has similar symmetries: $< \psi|U_T|\phi > = < \hat{\psi}|U_T|\hat{\phi} >$, provided either i) $\hat{\phi}(\theta) = \phi(\theta+\theta_o), \hat{\psi}(\theta) = \psi(\theta+\theta_o)$, for some $\theta_o \in [0, 2\pi]$; or ii) $\hat{\phi}(\theta) = \exp(in\theta)\phi(\theta), \hat{\psi}(\theta) = \exp(in\theta - in^2\omega T/2)\psi(\theta - n\omega T)$, for some integer n.

Now suppose, for contradiction, that there were some measure μ on $X \times X$ giving rise to these transition elements. It follows from these symmetries that μ must be invariant under i) simultaneous rotations of both circles through any angle $\theta_o \in [0, 2\pi]$; and ii) rotation of the second circle through angle $n\omega T$, and simultaneous multiplication of μ by the phase factor $e^{i(-n\theta_1 + n\theta_2 - n^2\omega T/2)}$. Hence, the absolute value of this measure, $|\mu|$, must be invariant under i) simultaneous rotations of both circles through the same angle; and ii) rotation of the second circle through angle $n\omega T$, with no rotation of the first circle. Now choose[7] the time T such that ωT is not a rational multiple of 2π. Then the rotations in ii) above are dense among all rotations of the circle. Hence[8], for any such choice of T, $|\mu|$ must be invariant under arbitrary rotations of

n. That is, $G(\theta_1, \theta_2)$ should be the result of "wrapping" about the circle the Green's function for a free particle on the line. But this sum above fails to converge (absolutely).

[7]It is not clear whether or not there exists a measure reproducing the transition amplitudes for the exceptional T-values. But note that, for ωT an integral multiple of 4π, U_T is the identity mapping, and so, for this choice, there *does* exist a suitable measure on $X \times X$.

[8]Here is a direct argument. It suffices to prove that $|\mu|$ is proportional to Lebesque measure when applied to any rectangle, $A = (0, a) \times (0, b)$. Let n be any positive integer, and set $B = (0, 1/n) \times (0, 1/n) \subset X \times X$. Then A can be covered by not more than $(an+1)(bn+1)$ images of B under the rotations given in i) and ii) above; and within A there can be contained at least $(an-1)(bn-1)$ disjoint such images of B. It follows that $(an-1)(bn-1)/(2\pi n+1)^2 \leq |\mu|(A)/|\mu|(X \times X) \leq (an+1)(bn+1)/(2\pi n-1)^2$. Let $n \to \infty$.

the two circles individually, i.e., $|\mu|$ must be a constant multiple of Lebesque measure on the torus. But this implies that μ is some multiple, $G(\theta,\theta')$, of Lebesque measure on the torus, with $|G(\theta,\theta')| = const$. That is to say, this implies there is a bounded Green's function for the time evolution of this system. But, as we have noted above, there can be no such Green's function.

Appendix B - Measures

In this appendix, we organize the subject of (complex-valued) measures in a way convenient for application to path integrals and other physical situations.

Fix a set X.

A *premeasure* on X consists of a nonempty collection, $\mathcal{M}$, of subsets of X, together with a mapping, $\mathcal{M} \xrightarrow{\mu} C$, from $\mathcal{M}$ to the complex numbers, satisfying the following two conditions:

1. For any $A, B \in \mathcal{M}$, we have $A - B \in \mathcal{M}$ and $A \cup B \in \mathcal{M}$.

2. For any $A, B \in \mathcal{M}$, with $A \cap B = \emptyset$, we have $\mu(A \cup B) = \mu(A) + \mu(B)$.

It follows immediately from condition 1 that $\emptyset \in \mathcal{M}$, and that any result of taking finite unions, intersections, and differences of sets in $\mathcal{M}$ is again in $\mathcal{M}$. It also follows that any finite union of sets in $\mathcal{M}$ is a finite union of disjoint sets in $\mathcal{M}$. It follows immediately from condition 2 that $\mu(\emptyset) = 0$, and that μ is additive on any finite collection of disjoint elements of $\mathcal{M}$. Note that "∞" is not allowed as a value for μ. These conditions are, in some sense, the minimum in order that $\mathcal{M}, \mu$ be regarded as at all "measure-like". Think of a premeasure as having all the properties of a full-fledged measure, but restricted to *finite* operations (sums, unions, etc). Of course, every measure (as defined, e.g., in Halmos) gives rise to a premeasure. Premeasures arise frequently in physical situations, a common example being the matrix elements of a bounded operator (Sect. 2).

Here are two examples. For the first, let X be any infinite set, let $\mathcal{M}$ consist of all subsets of X that are either finite, or cofinite (i.e., having finite complement in X). Let, for $A \in \mathcal{M}$, $\mu(A)$ be the number of elements in A if A is finite, and minus the number of elements in the complement of A if A is cofinite. This is a premeasure on X. For the second example, let X be the interval $(0, 1)$, and let $\mathcal{M}$ consist of all finite unions of disjoint intervals (open, closed, or half-open) in X. Fix a complex number c. For $A \in \mathcal{M}$, let $\mu(A)$ be the sum of the lengths of those intervals, plus c in case one of those intervals extends to 0 (i.e., is of the form $(0, a)$ or $(0, a]$). This is a premeasure on X. (It is basically Lebesque measure, except that there is additional measure, of strength c, "concentrated at 0"; this despite the fact that $0 \notin X$.)

While the conditions for a premeasure are simple and natural, these conditions are just too weak to have mathematically useful consequences. For example, premeasures support only a very rudimentary integration theory, as we shall see shortly. We now introduce two additional, stronger conditions that may be imposed on a premeasure.

Fix a premeasure, $\mathcal{M}, \mu$ on X. For any $A \in \mathcal{M}$, set

$$|\mu|(A) = \operatorname{lub} \sum_{i=1}^{s} |\mu(A_i)|, \tag{16}$$

where $A_1, \cdots, A_s \in \mathcal{M}$ is any finite collection of disjoint subsets of A, and the least upper bound is over all such collections (for all values of the positive integer s). Thus, $|\mu|(A) \geq 0$, with "∞" a possible value. We also have $|\mu|(A) \geq |\mu(A)|$. [Note that had we allowed, on the right side in the formula above, infinite collections of disjoint $A_i \in \mathcal{M}$, the result would have been the same number, $|\mu|(A)$. Had we allowed just a single $A_1 \in \mathcal{M}$, the result would have been a number between $|\mu|(A)/\pi$ and $|\mu|(A)$.[9]] We say the premeasure $\mathcal{M}, \mu$ has *bounded variation* if $|\mu|(A)$ is finite for every $A \in \mathcal{M}$. For instance, for $\mathcal{M}, \mu$ any premeasure we have, setting $\hat{\mathcal{M}} = \{A \in \mathcal{M} | \ |\mu|(A) < \infty\}$, that both $\hat{\mathcal{M}}, \mu$ and $\hat{\mathcal{M}}, |\mu|$ are premeasures having bounded variation. In particular, whenever $\mathcal{M}, \mu$ itself already has bounded variation, then $\mathcal{M}, |\mu|$ is also a premeasure having bounded variation.

A premeasure, $\mathcal{M}, \mu$ on X is said to have the *nesting property* provided: Given any $A_1 \supset A_2 \supset \cdots \in \mathcal{M}$ with $\cap A_i = \emptyset$, we have $\mu(A_i) \to 0$ as $i \to \infty$. In essence, the nesting property requires that the complex number $\mu(A)$ be "continuous in the set A". For instance, the nesting property is equivalent to each of the following: i) Given $A_1 \supset A_2 \supset \cdots \in \mathcal{M}$, with $A = \cap A_i \in \mathcal{M}$, then $\mu(A_i) \to \mu(A)$ as $i \to \infty$. ii) Given $A_1 \subset A_2 \subset \cdots \in \mathcal{M}$ with $A = \cup A_i \in \mathcal{M}$, then $\mu(A_i) \to \mu(A)$ as $i \to \infty$. iii) Given disjoint $A_1, A_2, \cdots \in \mathcal{M}$, with $A = \cup A_i \in \mathcal{M}$, then $\sum \mu(A_i) = \mu(A)$.

These two conditions on a premeasure — bounded variation and the nesting property — are clearly closely related to each other. For instance, in case

[9] This is a consequence of the following fact: Given any finite collection, $c_1, c_2, \cdots, c_s$ of complex numbers, $\operatorname{lub}_{S \subset \{1,2,\cdots,s\}} |\sum_{i \in S} c_i| \geq (1/\pi) \sum_{i=1}^{s} |c_i|$.

μ happens to take all its values in the nonnegative reals, then the nesting property implies bounded variation (but not conversely, as is seen by setting $c = 1$ in an example above). But in the full complex case, these two properties are logically distinct. Indeed, the first of the examples above satisfies the nesting property but, provided the set X is uncountable, fails to have bounded variation; while the second example has bounded variation but not the nesting property. The close relationship between the two can be seen in the following. Let $X, \mathcal{M}, \mu$ be a premeasure, having both bounded variation and the nesting property. Then, we claim, $\mathcal{M}, |\mu|$ also has both bounded variation and the nesting property. To prove this, we have only to show: Given $A_1 \supset A_2 \supset \cdots \in \mathcal{M}$, with $\cap A_i = \emptyset$, then $|\mu|(A_i) \to 0$ as $i \to \infty$. To see this, suppose, for contradiction, that instead $|\mu|(A_i) \to r > 0$. By deleting the first few A_i, we may assume $|\mu|(A_1) \leq 6r/5$. Choose $B \in \mathcal{M}$, $B \subset A_1$, with $|\mu(B)| \geq r/4$. Then we have $\cap(B \cap A_i) = \emptyset$, while $|\mu(B \cap A_i)| \geq |\mu(B)| - |\mu|(A_1 - A_i) \geq r/20$. This contradicts the nesting property of $\mathcal{M}, \mu$.

A premeasure, $\mathcal{M}, \mu$, on X is called a *measure* provided i) it has bounded variation, ii) it satisfies the nesting property, and iii) it also satisfies the following condition: Given any $A_1 \subset A_2 \subset \cdots \in \mathcal{M}$, with $|\mu|(A_i)$ bounded, then $\cup A_i \in \mathcal{M}$. Thus, in order that a premeasure be a measure, the function μ must be "well-behaved", in the sense of condition i) and ii) above, and further have $\mathcal{M}$ include "as many sets as feasible", in the sense of condition iii). Note that this condition iii) cannot be strengthened to demand still more sets in $\mathcal{M}$, for $\cup A_i$ can never be in $\mathcal{M}$ when the $|\mu|(A_i)$ are unbounded, by condition i). For $\mathcal{M}, \mu$ a measure, it follows that each of the following is also in $\mathcal{M}$: any countable intersection of sets in $\mathcal{M}$; any countable union of sets in $\mathcal{M}$, if all are subsets of some common element of $\mathcal{M}$; and any countable union of sets $A_i \in \mathcal{M}$, provided $\sum |\mu|(A_i) < \infty$. This is the standard notion of a "measure" (adapted to complex-valued) in the textbooks (ref). Note, e.g., that, for $\mathcal{M}, \mu$ a measure on X, then $\mathcal{M}, |\mu|$ is also a measure on X.

It will turn out that the best way to generate measures is by extending premeasures. To explain this requires a definition. Let $\mathcal{M}, \mu$ be a premeasure on X. An *extension* of $\mathcal{M}, \mu$ is a premeasure $\mathcal{M}', \mu'$ on X such that $\mathcal{M}' \supset \mathcal{M}$ and $\mu' = \mu$ on $\mathcal{M}$. Note, e.g., that an extension of an extension is an extension of the original premeasure. We note that having bounded variation and having the nesting property are passed backward under extensions. That is, if an extension $\mathcal{M}', \mu'$ of $\mathcal{M}, \mu$ has of bounded variation (resp., has the nesting property), then $\mathcal{M}, \mu$ itself must have bounded variation (resp, the nesting property). But this does not work in the other direction. Thus, a necessary condition that premeasure $\mathcal{M}, \mu$ have an extension that is a measure is that $\mathcal{M}, \mu$ have bounded variation and the nesting property.

So, what are the possibilities for extending a given premeasure? We first remark that *every* premeasure on X, save the case in which $\mathcal{M}$ consists already of all subsets of X, admits a proper extension [10] But, unfortunately, such extensions are not very useful, for two reasons. First, these extensions generally involve making arbitrary choices, e.g., of the values μ' assigns to the new subsets. Unless those choices are based on physical considerations, the resulting extension is likely to be unphysical. Second, there is no guarantee that, when $\mathcal{M}, \mu$ has bounded variation or the nesting property, then these extensions will retain these properties.

But it turns out that there is a way, whenever the given premeasure $\mathcal{M}, \mu$ happens to have both bounded variation and the nesting property, to construct certain extensions that avoid these difficulties: These extensions are based *solely* on the information contained already in $\mathcal{M}, \mu$ (thus, introducing no arbitrary, unphysical elements), and automatically retain bounded variation and the nesting property. This construction proceeds as follows. Let premeasure $\mathcal{M}, \mu$ on X have bounded variation and the nesting property. Let $Y \subset X$ be any subset with the following property: There exists a nested collection, $A_1 \subset A_2 \subset \cdots \in \mathcal{M}$ with $Y = \cup A_i$ and the $|\mu|(A_i)$ bounded. This property requires, roughly, that Y be expressible as a countable union of elements of $\mathcal{M}$ which do not get "too large", as measured by $|\mu|$. We now construct an extension of $\mathcal{M}, \mu$ which "includes Y". [Note that, were $\mathcal{M}, \mu$ already a measure, then we would automatically have $Y \in \mathcal{M}$. In this case, the extension below would simply recover $\mathcal{M}, \mu$.] Let $\mathcal{M}'$ consist of $\mathcal{M}$, together with, for all $A \in \mathcal{M}$, the sets $A \cap Y$, $A \cup Y$, $A - Y$, $Y - A$, and $(A - Y) \cup (Y - A)$. Then $\mathcal{M}'$ satisfies the first condition for a premeasure.

[10] The proof is along the following lines. Let $Y \subset X$ with $Y \notin \mathcal{M}$. Let $\mathcal{M}'$ consist of the sets in $\mathcal{M}$, together with, for every $A \in \mathcal{M}$, the sets $Y - A, A - Y, A \cup Y, A \cap Y$, and $(A - Y) \cup (Y - A)$. Then $\mathcal{M}'$ satisfies the first condition for a premeasure. We must now extend $\mathcal{M} \xrightarrow{\mu} C$ to $\mathcal{M}' \xrightarrow{\mu'} C$, while retaining the second condition for a premeasure. To do this, first extend μ, arbitrarily, to some new set in $\mathcal{M}'$; then to those additional sets in $\mathcal{M}'$ on which its value is thereby determined via condition ii) for a premeasure; then, again arbitrarily, to some new set in $\mathcal{M}'$, etc. Continue in this way (using Zorn's lemma) until $\mathcal{M}'$ is exhausted.

Define $\mathcal{M}' \xrightarrow{\mu'} C$ as follows: First, set $\mu' = \mu$ on $\mathcal{M}$. Then, for any $A \in \mathcal{M}$, set $\mu'(A \cap Y) = \lim \mu(A \cap A_i)$, $\mu'(A \cup Y) = \lim \mu(A \cup A_i)$, and similarly for $A - Y$, $Y - A$, and $(A - Y) \cup (Y - A)$. The limits (as $i \to \infty$) on the right exist, by bounded variation of $\mathcal{M}, \mu$, and are independent of the choice of A_i, by the nesting property. This function μ' on $\mathcal{M}'$ satisfies the second condition for a premeasure, as follows immediately from the corresponding condition on μ. So, $\mathcal{M}', \mu'$ is a premeasure space. It further follows that this new premeasure has both bounded variation and the nesting property. Indeed, we have $|\mu'| = |\mu|$ on $\mathcal{M}$, while $|\mu'|(A \cap Y) = lim|\mu|(A \cap A_i)$, etc. Bounded variation and the nesting property for $\mathcal{M}', \mu'$ now follow from the corresponding properties for $\mathcal{M}, \mu$.

We have seen that, given a premeasure $\mathcal{M}, \mu$, a necessary condition that it have an extension that is a measure is that $\mathcal{M}, \mu$ have bounded variation and the nesting property. A key theorem of this subject is the converse of this: Every premeasure having both bounded variation and the nesting property has an extension to a measure. In other words, the third condition in the definition of a measure — that $\mathcal{M}$ be "sufficiently large" — can always be achieved by an extension. Furthermore, this extending measure is unique in a certain sense.

Theorem 2.[11] Let, on X, $\mathcal{M}, \mu$ be a premeasure having both bounded variation and the nesting property. Then there exists a unique measure, $\mathcal{M}', \mu'$, on X that is an extension of $\mathcal{M}, \mu$, and is minimal in the following sense: Any other measure, $\mathcal{M}'', \mu''$, on X that is an extension of $\mathcal{M}, \mu$ is also an extension of $\mathcal{M}', \mu'$.

The proof is actually quite simple. [???] Denote by $\hat{\mathcal{M}}$ the collection of subsets of X obtained by intersecting all collections closed under differences and countable intersections and unions. Consider (premeasure) extensions, $\mathcal{M}', \mu'$, of $\mathcal{M}, \mu$, with $\mathcal{M}' \subset \hat{\mathcal{M}}$, ordered by inclusion. By Zorn's lemma, there is a maximal element. Were this element not a measure, then we

[11] There is an immediate generalization of all this, in which the premeasure is valued, instead of in the complexes, in an arbitrary complete abelian topological group G. Then bounded variation is stated thus: For every $A \in \mathcal{M}$, $\{\mu(B) | B \in \mathcal{M}, B \subset A\}$ has compact closure in G. The definitions of the nesting property and of an extension go through unchanged. Finally, a premeasure is a measure if, whenever $A_1, A_2, \cdots \in \mathcal{M}$ and the subset $\{\mu(B) | B \in \mathcal{M}, B \subset \cup A_i\}$ of G has compact closure, then $\cup A_i \in \mathcal{M}$. With these definitions, the theorem again holds.

would be able to apply the construction above, adding a new set $Y \subset X$ to it.

A key reason for having (pre)measures at all is that they permit the carrying out of integrals. We now briefly discuss the status of integration, and its relation to the conditions above.

Fix a premeasure, $\mathcal{M}, \mu$ on X. A set $B \subset X$ will be called *measurable* if $B \cap A \in \mathcal{M}$ for every $A \in \mathcal{M}$. Thus, the complement of a measurable set, as well as any finite (but not, in general, infinite) union or intersection of measurable sets, yields a measurable set. A complex-valued function $X \xrightarrow{f} C$ will be called *measurable* if, inverse images, under f, of open sets in C are measurable. It follows from this that $f^{-1}[U]$ is measurable also for U closed, and, indeed, for U any set obtained by finite unions or intersections of open and closed sets. Note that the sum of two measurable functions need not be measurable in general. A *step function* is a measurable function with finite range. Thus, every step function f is of the form $f = a_1\chi_{A_1} + \cdots + a_s\chi_{A_s}$, where $a_i \in C$, the A_i are measurable subsets of X, and χ_{A_i} denotes the characteristic function of $A_i \subset X$. Conversely, any function of this form is a step function. Note that any finite linear combination of step functions is again a step function. Were $\mathcal{M}, \mu$ a full measure on X, then, by virtue of condition iii), countable unions and intersections of measurable sets would be measurable, and finite linear combinations of measurable functions would be measurable.

Again, $\mathcal{M}, \mu$ is a premeasure on X. There is a natural notion, in this space, of the integral of any step function over any set $A \in \mathcal{M}$. This integral is given by the following formula:

$$\int_A f \; d\mu = \sum_{i=1}^{s} a_i \mu(A_i \cap A), \tag{17}$$

where f has been expanded, in terms of characteristic functions, as above[12]. Note that the right side is linear in the function f (i.e., is such that $\int_A (f + cg) = \int_A f + c\int_A g$), and is additive in the set A (i.e., is such that, for $A \cap B = \emptyset$, $\int_{A \cup B} f = \int_A f + \int_B f$). In short, there is, even in a mere premeasure

[12]A similar integral can be defined in the case in which the measure is valued in a locally compact abelian topological group, and the function to be integrated is valued in (in general, some different) locally compact abelian topological group. Then the integral, above, is valued in the tensor product of the two groups.

space, a certain, natural notion of "integration". But, unfortunately, the only functions that can be integrated via Eqn. (17) are the step functions, and these form far too small a class to be the basis for a viable integration theory. After all, few of the functions that arise in physical applications are step functions.

Let us now impose on the premeasure $\mathcal{M}, \mu$ the further condition that it have bounded variation. Then, for any $A \in \mathcal{M}$ and any step function f, we have

$$|\int_A f\ d\mu| \leq \mathrm{lub}_A|f|\ |\mu|(A), \tag{18}$$

as follows immediately from Eqn. (17). Next let $f_1, f_2, \cdots$ be any sequence of step functions, converging uniformly on $A \in \mathcal{M}$ to some complex-valued function f on A. Then, as follows directly from Eqn. (18), the complex numbers $\int_A f_i$ also converge to some complex number, and that number depends only on f, and not on the choice of the f_i. We write this number $\int_A f\ d\mu$, and call it the integral of f over $A \in \mathcal{M}$. The integral $\int_A f\ d\mu$ so defined has the expected properties: It is linear in the function f, and (finitely) additive in the set A. Thus, in the case in which our premeasure has bounded variation, we may integrate over any set in $\mathcal{M}$ any function that is a uniform limit of step functions. In this way, we greatly expand the collection of functions that we can integrate, resulting in a far richer integration theory. For example, any bounded, measurable function (*not* necessarily a step function) can now be integrated over any $A \in \mathcal{M}$, for every such function is a uniform limit, on A, of step functions. Indeed, the class of functions on A that are integrable in this way is a rather large one: While all uniform limits of step functions are bounded, some, in general, are not even measurable. As an example of this integration theory, let $X = (0, 1)$, and let the premeasure be that given above, with "additional measure of magnitude c located at 0". This premeasure, as already noted, has bounded variation. Then the function f given by $f(x) = 1 - x$ is integrable over $A = X$, and $\int_A f\ d\mu = 1/2 + c$.

But even this integration theory — for a premeasure having bounded variation — is not as rich as one might like. For instance, in the example above, the given function f is not integrable over the set $A \subset X$ consisting of the rationals, for $A \notin \mathcal{M}$. Furthermore, setting $A_1 = [1/2, 1), A_2 = [1/4, 1/2), \cdots$, so the A_i are disjoint and have $\cup A_i = X$, it is false that $\sum \int_{A_i} f\ d\mu = \int_A f\ d\mu$, for the left side has value $1/2$, the right, $1/2+c$. These defects are "corrected" in the integration theory of a full measure space, as we now discuss.

Now let $\mathcal{M}, \mu$ be a measure on X. Fix a measurable function f on X. Denote by $\hat{\mathcal{M}}$ the collection of all subsets $A \in \mathcal{M}$ on which f is bounded. For any $A \in \hat{\mathcal{M}}$, set $\hat{\mu}(A) = \int_A f \; d\mu$, noting that f is necessarily a uniform limit of step functions on A, and so the right side makes sense, by Eqn. (18). This $\hat{\mathcal{M}}, \hat{\mu}$, we claim, is a premeasure on X. This is immediate from the fact that the integral is additive in the set A. Furthermore, we have $|\hat{\mu}(A)| \leq lub_A |f| \; |\mu|(A)$. It follows from this that $|\hat{\mu}|(A) \leq lub_A |f| \; |\mu|(A)$; and from this in turn that the premeasure $\hat{\mathcal{M}}, \hat{\mu}$ is of bounded variation, and satisfies the nesting property. Now apply, to the premeasure $\hat{\mathcal{M}}, \hat{\mu}$, Theorem 2, to obtain the unique measure minimally extending it. The result defines the *integral* of f: The sets in this extension are those over which the integral of f converges; and the value of the measure in this extension is the value of the integral of f over the set. The important properties of integrals now merely repeat properties of measures.