Mathematics

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The currency of mathematics is what are called *assertions*. An assertion is a precise, unambiguous, mathematical statement to the effect that something is true (e.g., that some relation between mathematical objects holds; that some object has some property; etc). It isn't necessary, in order to qualify an assertion, that the statement actually *be* true. But it should be the case that the statement is sufficiently clear that it is destined, ultimately, to be found either true or false (as opposed to being vague, or arguable as to meaning). Thus, the phrases that go into assertions are typically those from mathematics, e.g., "there exists one and only one"; "for no x does"; "the collection of all y such that"; "has the property"; "if two mappings from X to Y both satisfy"; "then f is discontinuous on at least seven points of X". Here, for contrast, are some examples of phrases that are rarely found in assertions: "we can find an x with"; "the function f is ambiguous for $x = 6$"; "we cannot add these elements"; "you just do ... and it gives you that"; "we can write"; "Eqns. (6) and (9) are inconsistent".

In principle, *all* the conditions, hypotheses and assumptions that go into the assertion are to be included as part of it. In practice, however, some conditions are often omitted. Typically, these are conditions that are understood to remain in force, either because they are universally in force for the entire discussion, or are still in force from a recent assertion. Assertions work best if they are single sentences. But, unfortunately this is sometimes not practicable, usually because there are so many underlying conditions that, were they all packed into a single sentence, the result would be difficult to read. Then two or more sentences (so organized that it is clear that they are to be taken together to form a single assertion) are necessary. It is regarded as bad form to include, in an assertion, other things that are not part of that assertion, e.g., how you feel about the statement, how you arrived at it, etc. Such information can of course be expressed, but this is usually done in other sentences. Assertions should be as brief as they can be, without sacrificing clarity.

The words contained in assertions are, generally of two types: the ordinary words of mathematics ("such that", "implies", "exists", "unique", "not", etc), together with various *defined terms*. A definition in mathematics merely introduces one or more words to stand for a longer string of words: It is a shortening of the language. [Note that this usage is somewhat different from that of the definitions that appear in the dictionary. In retrospect, perhaps it would have been advisable for mathematics to have chosen a word other than "definition".] Here is an example of a mathematical definition: "A *twinned prime* is a pair of prime integers that differ by 2." We don't want to have to repeat "a pair of prime integers that differ by 2" all the time, so we shorten it. It is the intention that, from this definition, you can decide, without doubt or ambiguity, whether or not a given mathematical object qualifies as a "twinned prime". Note that definitions are not assertions: They do not say that anything having truth value, but rather merely allow some words to be replaced by others. Of course, a proper definition of "blah" should give an unambiguous rule for what mathematical objects will, and what will not, be called blahs. The definition should contain no extraneous material, and should be as brief as it can be without sacrificing clarity.

Here are some examples of assertions:

All prime numbers are odd. This is an assertion. In fact, it is false: A counterexample is the integer "2".

In order to integrate the right side of Eqn. (8), you need boundary conditions. This is not an assertion. We don't know the mathematical content of "in order to integrate" and "need boundary conditions". **There exists a continuous function f which, when inserted into the right side of Eqn. (8), yields a function whose integral over R^3 diverges.** This is an assertion. It may, for example, be what was meant in the previous example. [That the function f is to be, say, real-valued presumably remains in force from the discussion that preceeded this assertion.]

We cannot solve Eqn. (6). This is not an assertion. What does "solve" mean? What does "cannot" mean (inability, or impossibility)? **There exists a continuous function g for which there is no function f satisfying Eqn. (6).** This is an assertion.

To get a group, you can just say ... This is not an assertion: What does it mean to "get" a group? What is the mathematical content of "saying" something? **This set G, with the product-operation given by Eqn. (9), is a group.** This is an assertion.

There are at least three contexts in mathematics in which assertions play a major role.

The first is in informal discussions, both oral and written. Frequently, an issue under discussion can be clarified — and thus the entire discussion made more pointed and productive — by introducing an assertion to replace some rather vague ideas. Thus, one might replace the idea "Is continuity of f needed for this argument?" by "Is the assertion 'There exists a discontinuous function f having the properties ...' true or false?" Of course, not everything in mathematics — and certainly not everything in physics — can be rendered as an assertion. But, as a general rule, the more assertion-filled is one's conversation, the better.

The second is in the statement of a theorem. The statement of a theorem is the prototype of an assertion (the rule being that, if an assertion is labeled a "theorem", then you are saying that it is actually true). Note, therefore, that all the conditions and hypotheses are, in principle at least, to be included in the statement of the theorem (although, again, conditions held in force from earlier discussion may be omitted). Note also that the statement of a theorem is not to contain any extraneous information. Here is an example of a theorem: "The number e is irrational." [Presumably, "e" and "irrational" were previously defined, e.g., by "$e = 1 + 1/1! + 1/2! + \cdots$" and "is not equal to a quotient of integers", respectively.]

The third is in the proof of a theorem. Generally speaking, a proof is organized to consist of a sequence of assertions, each of which usually has attached to it some words indicating how that assertion follows from the hypothesis and earlier assertions. Thus, each assertion in this sequence is true, and can be seen to be true already at that point of the proof. These assertions represent "steps" — and the more nearly these steps are equal in difficulty, the better crafted your proof is. Proofs often also contain some "organizing" phrases or sentences, examples of which include: "for contradiction", "we first show", "it is convenient to consider first the special case", "consider the expression", "there remains only", etc.

The proof should actually demonstrate that the assertion that constitutes the statement of the theorem is true. Thus, to prove "This set G, with the product-rule of Eqn. (9), is a group.", you would normally check that that set with that product rule satisfies each of the conditions for being a group. [Checking additional conditions is bad form.] To prove "Every such f is continuous.", you would normally consider any such f, without further conditions on that f, and demonstrate continuity. To prove "There exists a $x \in X$ that is irrational.", you would normally display a particular x that is in fact in X, and check that that x is indeed irrational. [You don't have to find them all; and in fact one normally displays more than one x only if it illustrates something interesting.] To prove "No prime number p has the property mwumf.", you would normally assume, for contradiction, that a prime number p does have property mwumf, and arrive from this at a contradiction. In the discussion above, we inserted the word "normally" for good reason: There exist other methods of proof (although they are somewhat less common) than those suggested in these examples.

Here is an example of a proof, of the theorem "The number e is irrational." above. "Let, for contradiction, $1 + 1/1! + 1/2! + \cdots = n/m$, where n and m are integers. Multiply both sides by $m!$, to obtain $[m! + m!/1! + m!/2! + \cdots + m!/m!] + [1/(m+1) + 1/((m+1)(m+2)) + \cdots] = n(m-1)!$. But the right side, and the first square bracket on the left side, are both integers, while the second square bracket on the left side, being greater than 0 and less than $1/(m+1) + 1/(m+1)^2 + \cdots = 1/m$, is not. This is a contradiction."