Definitions

1 Manifolds

Let m and n be non-negative integers, and V and open subset of R^m. Then a mapping $V \xrightarrow{\kappa} R^n$ is said to be **smooth** provided all partial derivatives of all orders of the corresponding n functions of m variables exist and are continuous.

Let M be a set, and n a non-negative integer. An n**-chart** on M is a subset U of M, together with a mapping $U \xrightarrow{\phi} R^n$, such that i) the subset $\phi[U]$ of R^n is open, and ii) the mapping ϕ is one-to-one.

Let M be a set and n a non-negative integer; and let (U, ϕ) and (U', ϕ') be two n-charts on this set. These two charts are said to be **compatible** provided i) the subsets $\phi[U \cap U']$ and $\phi'[U \cap U']$ of R^n are both open, and ii) the maps $\phi[U \cap U'] \xrightarrow{\phi' \circ \phi^{-1}} R^n$ and $\phi'[U \cap U'] \xrightarrow{\phi \circ \phi'^{-1}} R^n$ are both smooth.

An n**-manifold**, where n is a non-negative integer, is a set M, together with a collection $\mathcal{C}$ of n-charts on M, such that i) any two charts in the collection $\mathcal{C}$ are compatible, ii) the U's of the charts in $\mathcal{C}$ cover M, and iii) every n-chart compatible with all the charts in $\mathcal{C}$ is itself in the collection $\mathcal{C}$.

Note: Condition iii) is sometimes omitted, in light of the following fact: Given set M and collection $\mathcal{C}$ of n-charts satisfying i) and ii) above, then $(M, \hat{\mathcal{C}})$, where $\hat{\mathcal{C}}$ is the collection of all n-charts on M compatible with all those in $\mathcal{C}$, satisfies all three conditions above, i.e., is a manifold as here defined. Let us agree that, hereafter, whenever we say "n-manifold", then "n is a non-negative integer" is implied.

Let $(M, \mathcal{C})$ be an n-manifold. A **(smooth) function** on M is a map $M \xrightarrow{f} R$ with the following property: For every chart (U, ϕ) in the collection $\mathcal{C}$, the map $\phi[U] \xrightarrow{f \circ \phi^{-1}} R$ is smooth.

Note: Each constant function on M is smooth, as is the pointwise sum and product of any two smooth functions.

Let $(M, \mathcal{C})$ be an n-manifold. A **(smooth) curve** on M is a map $R \xrightarrow{\gamma} M$ with the following property: For every chart (U, ϕ) in the collection $\mathcal{C}$, i) $\gamma^{-1}[U]$ is open in R, and ii) the mapping $\gamma^{-1}[U] \xrightarrow{\phi \circ \gamma} R^n$ is smooth.

Let $(M, \mathcal{C})$ be an n-manifold, and let $p \in M$. A **tangent vector** at p is a mapping $\mathcal{F} \xrightarrow{\xi} R$, where $\mathcal{F}$ denotes the collection of all smooth functions on this manifold, such that: i) for f any constant function, $\xi(f) = 0$; and for f and g any two smooth functions, ii) $\xi(f + g) = \xi(f) + \xi(g)$, and iii) $\xi(fg) = f(p)\xi(g) + g(p)\xi(f)$.

Note: The set of tangent vectors at p has the structure of an n-dimensional vector space (under the obvious definitions of addition and scalar multiplication of tangent vectors).

Let $(M, \mathcal{C})$ be an n-manifold, $R \xrightarrow{\gamma} M$ a smooth curve on this manifold, and λ_o a real number. Set $p = \gamma(\lambda_o) \in M$. The **tangent** to γ at λ_o is the tangent vector ξ at p such that: For every smooth function f on M, $\xi(f) = d(f \circ \gamma)/d\lambda \mid_{\lambda_o}$ (noting that the right side does indeed define a tangent vector).

Let $(M, \mathcal{C})$ be an n-manifold, $p \in M$, ξ a tangent vector at p, and (U, ϕ) an n-chart in the collection $\mathcal{C}$, such that $p \in U$. The **components** of ξ with respect to this chart are the n numbers $(\xi^1, \cdots, \xi^n)$ such that: For any smooth function f on M,

$$\xi(f) = \Sigma_{i=1}^n \; \xi^i \; [\partial(f \circ \phi^{-1})/\partial x^i] \mid_{\phi(p)}. \tag{1}$$

Let $(M, \mathcal{C})$ be an n-manifold, and let $p \in M$. A **covector** at p is a linear mapping $T \xrightarrow{\mu} R$, where T is the n-dimensional vector space of tangent vectors at p.

Note: The set of covector at p has the structure of an n-dimensional vector space (under the obvious definitions of addition and scalar multiplication of covectors).

Let $(M, \mathcal{C})$ be an n-manifold, let f be a smooth function on M, and let $p \in M$. The **gradient** of f at p is the covector, written ∇f, at p with the following action: For any tangent vector ξ at p, $(\nabla f)(\xi) = \xi(f)$ (noting that the right side does indeed define a covector).

Let $(M, \mathcal{C})$ be an n-manifold, $p \in M$, μ a covector at p, and (U, ϕ) an n-chart in the collection $\mathcal{C}$, such that $p \in U$. The **components** of μ with respect to this chart are the n numbers $(\mu_1, \cdots, \mu_n)$ such that: For any tangent vector ξ at p,

$$\mu(\xi) = \Sigma_{i=1}^{n} \mu_i \, \xi^i, \tag{2}$$

where $(\xi^1, \cdots, \xi^n)$ are the components of ξ with respect to this chart.

Let $(M, \mathcal{C})$ be an n-manifold. A **tangent vector field** (resp, **covector field**) on M is a mapping that sends each point p of M to a tangent vector (resp, covector) at p. Fix such a field ξ (resp, μ), and let (U, ϕ) be an n-chart in the collection $\mathcal{C}$. Consider the following map from the open set $\phi[U]$ in R^n to R^n: It sends $(x^1, \cdots, x^n) \in \phi[U]$ to the components of ξ (resp, μ), evaluated at $\phi^{-1}(x^1, \cdots, x^n)$. We say that the tangent vector field ξ (resp, covector field μ) is **smooth** if this mapping is smooth.

Let $(M, \mathcal{C})$ be an n-manifold, and f a smooth function on M. The **gradient** of f is the smooth covector field (also written ∇f) on M whose value at each $p \in M$ is the gradient of f at p.

Let $(M, \mathcal{C})$ be an n-manifold, and μ a smooth covector field on M. This field is said to be **exact** if there exists a smooth function f on M such that $\mu = \nabla f$.

2 Measure Theory

Let X be a set, and $X \xrightarrow{f}$ real-valued function on X. Then we say that $\Sigma_X f$ **converges** to $a \in R$ provided: Given any $\epsilon > 0$, there exists a finite Y, $Y \subset X$ such that, for any finite Y', $Y \subset Y' \subset X$, $|\Sigma_{Y'} f - a| \leq \epsilon$. Here, the sum of f over a finite set has the obvious meaning: the sum of the values of f on that set.

Note: This is what (absolute) convergence of a sum *really* is. We may replace the reals, above, by the complexes (or, indeed, by any abelian topological group). Note that we can, potentially, "sum" over even an uncountable set X! [If you know the definition of a *net*, you can formulate the above even more elegantly: Let the directed set be that of the finite subsets of X.]

A **measurable space** is a set S, together with a nonempty collection, $\mathcal{S}$ of subsets of S, satisfying the following two conditions:

1. For any A, B in the collection $\mathcal{S}$, the set[1] $A - B$ is also in $\mathcal{S}$.
2. For any $A_1, A_2, \cdots \in \mathcal{S}$, $\cup A_i \in \mathcal{S}$.

Note: Sometimes the additional condition $S \in \mathcal{S}$ is included in which case $(S, \mathcal{S})$ is called a $\sigma-$**algebra**.

Let $(S, \mathcal{S})$ be a measurable space. A **measure** on $(S, \mathcal{S})$ consists of a nonempty subset of $\mathcal{S}$, $\mathcal{M} \subset \mathcal{S}$, together with a mapping $\mathcal{M} \xrightarrow{\mu} R^+$ (where R^+ denotes the set of nonnegative reals), satisfying the following two conditions:

1. For any $A \in \mathcal{M}$ and any $B \subset A$, with $B \in \mathcal{S}$, we have $B \in \mathcal{M}$.
2. Let $A_1, A_2, \cdots \in \mathcal{M}$ be disjoint, and set $A = A_1 \cup A_2 \cup \cdots$. Then: This union A is in $\mathcal{M}$ if and only if the sum $\mu(A_1) + \mu(A_2) + \cdots$ converges; and when these hold that sum is precisely $\mu(A)$.

Fix any subset of the reals, $X \subset R$. Let $I_1, I_2, \cdots$ be any countable collection of open intervals (not necessarily disjoint) in R such that $X \subset I_1 \cup I_2 \cup \cdots$ (noting that there exists at least one such collection); and denote by m the sum of the lengths of these intervals (possibly ∞). By the **outer measure** of X, $\mu^*(X)$, we mean the greatest lower bound, over all such collections, of m. Thus, $\mu^*(X)$ is a non-negative number, or possibly "∞".

For X and Y be any two subsets of the reals, R, set $d(X, Y) = \mu^*(X - Y) + \mu^*(Y - X)$, so $d(X, Y)$ is a nonnegative or the symbol "∞".

Denote by $\mathcal{M}$ the collection of all subsets A of $S = R$ with the following property: Given any $\epsilon > 0$, there exists a $K \subset R$, where K is a finite union of open intervals, such that $d(A, K) \leq \epsilon$. And, for $A \in \mathcal{M}$, set $\mu(A) = \mu^*(A)$, a nonnegative number. A set $A \subset R$ is said to be **Lebesque measurable** if it is a countable union of sets in $\mathcal{M}$; and, for $A \in \mathcal{M}$, the number $\mu(A)$ is called the **Lebesque measure** of A.

[1]By $A - B$, we mean $A \cap B^c$, i.e., the set of all points of A that are not in B.

Let $(S, \mathcal{S})$ be a measurable space, and $S \xrightarrow{f} R$ a function on S. We say that this f is **measurable** if, for every open $O \subset R$, $f^{-1}[O] \in \mathcal{S}$.

Let $(S, \mathcal{S})$ be a measurable space. By a **step function** on S we mean a measurable, real-valued function on S with finite range.

Let $(S, \mathcal{S}, \mathcal{M}, \mu)$ be a measure space, with $S \in \mathcal{M}$, and let f be a step function (with values $a_1, \cdots, a_s$ on disjoint sets $A_1, \cdots, A_s \in \mathcal{M}$, where $A_1 \cup \cdots \cup A_s = S$) on S. The **integral** of f (over S, with respect to μ) is the number $\int_S f\, d\mu = a_1\, \mu(A_1) + \cdots a_s\, \mu(A_s)$.

Let $(S, \mathcal{S}, \mathcal{M}, \mu)$ be a measure space, with $S \in \mathcal{M}$, and let f be a bounded measurable function on S. Then the **integral** of f (over S, with respect to μ) is the number $\int_S f\, d\mu = \lim_{\epsilon \to 0} \int_S g_\epsilon\, d\mu$, where the g_ϵ are step functions such that $|f - g_\epsilon| \leq \epsilon$.

Let $(S, \mathcal{S}, \mathcal{M}, \mu)$ be a measure space (not necessarily finite), and let $f \geq 0$ be a measurable function (not necessarily bounded) on S. We say that f is **integrable** over $A \in \mathcal{S}$ provided there exist disjoint sets $A_1, A_2, \cdots$ each of finite measure and on each of which f is bounded, such that $\cup A_i = A$ and $\sum(\int_{A_i} f\, d\mu)$ converges. We write $\int_A f\, d\mu$ for that sum.

Let $(S, \mathcal{S}, \mathcal{M}, \mu)$ be a measure space, and f a measurable function (not necessarily nonnegative) on S. We say that f is **integrable** over $A \in \mathcal{S}$ provided each of the (nonnegative, measurable) functions $f^+ = (1/2)(|f| + f)$ and $f^- = (1/2)(|f| - f)$ is integrable over A. We write $\int_A f\, d\mu$ for the difference of these two integrals.

3 Hilbert Spaces

Recall that a **complex vector space** is a set H, together with a mapping $H \times H \xrightarrow{add} H$ (called **addition**, and written $x+y$) and a mapping $C \times H \xrightarrow{mult} H$ (called **scalar multiplication**, and written $c\, x$), satisfying the nine or so standard vector-space conditions. An **inner product** on a complex vector space H is a mapping $H \times H \xrightarrow{prod} C$ (called the **inner product**, and written

$\langle x \,|\, y\rangle$) which i) is antilinear in the first entry and linear in the second (i.e., satisfies $\langle x + c\,z \,|\, y\rangle = \langle x \,|\, y\rangle + \overline{c}\,\langle z \,|\, y\rangle$ and $\langle x \,|\, y + c\,z\rangle = \langle x \,|\, y\rangle + c\,\langle x \,|\, z\rangle$, for any $x, y, z \in H$ and $c \in C$); ii) complex-conjugates under order-reversal (i.e., satisfies $\langle x \,|\, y\rangle = \overline{\langle y \,|\, x\rangle}$ for any $x, y \in H$); and iii) is such that $\langle x \,|\, x\rangle \geq 0$, with equality if and only if $x = 0$). Fix any complex vector space H with inner product $\langle \ | \ \rangle$. Then, for $x \in H$, we set $\| \ x \ \| = \langle x \,|\, x\rangle^{1/2}$ (called the **norm** of x). A sequence $x_1, x_2, \cdots \in H$ is called a **Cauchy sequence** if, for every $\epsilon > 0$, there exists a number N such that $\| \ x_n - x_m \ \| \leq \epsilon$ whenever $n, m \geq N$. A sequence $x_1, x_2, \cdots \in H$ is said to **converge** to $x \in H$ provided $\| \ x - x_n \ \| \to 0$ as $n \to \infty$. A complex vector space with inner product is said to be **complete** if every Cauchy sequence therein converges to some element of H.

A **Hilbert space** is a complex vector space with inner product that is complete. [We frequently write $|x\rangle$ to indicate that x is an element of Hilbert space H.]

Let H be a Hilbert space. Element $|x\rangle$ is said to be **unit** provided $\| \ x \ \| = 1$; and elements $|x\rangle, |y\rangle$ are said to be **orthogonal** provided $\langle x \,|\, y\rangle = 0$. We say that $\sum |x_m\rangle$, where the sum is over $m = 0, 1, \cdots$, **converges** to $|x\rangle$ provided the partial sums, $\sum_1^n |x_m\rangle$, converge (as defined above) to x.

Let H be a Hilbert space. A collection $|x_\gamma\rangle$ (where γ ranges through some indexing set Γ) is called a **basis** for H provided i) each $|x_\gamma\rangle$ is unit; and any two $|x_\gamma\rangle$, for distinct γ, are orthogonal; and ii) for every $|x\rangle$ there exist complex numbers c_γ such that $\sum_\Gamma c_\gamma \, |x_\gamma\rangle$ converges to x.

Let $(S, \mathcal{S}, \mathcal{M}, \mu)$ be a measure space. By $L^2(S, \mathcal{S}, \mathcal{M}, \mu)$ we mean the Hilbert space whose elements are complex-valued, measurable, square-integrable functions f on S, with two functions identified if they differ on a set of measure zero; whose addition and scalar multiplication are pointwise addition and scalar multiplication of the functions; and whose innner product is given by $\langle f | g\rangle = \int_S \overline{f} g \, d\mu$ (noting that these operations are independent of representative; and that this last integral converges).

4 Operators

Let H be a Hilbert space. A (bounded) **operator** on H is a complex-linear map, $H \xrightarrow{A} H$, such that, for some number $a > 0$, $\parallel Ax \parallel \leq a \parallel x \parallel$ for every x. The greatest lower bound of such a's, written $|A|$, is called the **norm** of operator A.

Let H be a Hilbert space, and A a (bounded) operator on H. Let c be a complex number, and $|x\rangle$ a nonzero vector in H, such that $A|x\rangle = c\,|x\rangle$. Then we say that $|x\rangle$ is an **eigenvector** of A, with **eigenvalue** c. Fix c, and consider the collection consisting of all eigenvectors of A with eigenvalue c together with the zero vector. This collection is a subspace of H, called the **eigenspace** of A (corresponding to eigenvalue c).

A (bounded) operator A on H is said to be **self-adjoint** provided: For every $|x\rangle, |y\rangle \in H$, $\overline{\langle x|Ay\rangle} = \langle y|Ax\rangle$.

A (bounded) operator P on H is said to be a **projection** if P is self-adjoint and satisfies $P \circ P = P$.

A (bounded) operator U on H is said to be **unitary** if U is inner-product preserving (i.e., $\langle Ux|Uy\rangle = \langle x|y\rangle$, for every $|x\rangle, |y\rangle \in H$) and invertible (i.e., there exists a bounded operator V on H such that $U \circ V = V \circ U = I$).