Definitions

1 Manifolds

Let $m$ and $n$ be non-negative integers, and $V$ and open subset of $\mathbb{R}^m$. Then a mapping $V \xrightarrow{\phi} \mathbb{R}^n$ is said to be smooth provided all partial derivatives of all orders of the corresponding $n$ functions of $m$ variables exist and are continuous.

Let $M$ be a set, and $n$ a non-negative integer. An $n$-chart on $M$ is a subset $U$ of $M$, together with a mapping $U \xrightarrow{\phi} \mathbb{R}^n$, such that i) the subset $\phi[U]$ of $\mathbb{R}^n$ is open, and ii) the mapping $\phi$ is one-to-one.

Let $M$ be a set and $n$ a non-negative integer; and let $(U, \phi)$ and $(U', \phi')$ be two $n$-charts on this set. These two charts are said to be compatible provided i) the subsets $\phi[U \cap U']$ and $\phi'[U \cap U']$ of $\mathbb{R}^n$ are both open, and ii) the maps $\phi[U \cap U'] \xrightarrow{\phi'} R^n$ and $\phi'[U \cap U'] \xrightarrow{\phi^{-1} \circ \phi'} R^n$ are both smooth.

An $n$-manifold, where $n$ is a non-negative integer, is a set $M$, together with a collection $\mathcal{C}$ of $n$-charts on $M$, such that i) any two charts in the collection $\mathcal{C}$ are compatible, ii) the $U$'s of the charts in $\mathcal{C}$ cover $M$, and iii) every $n$-chart compatible with all the charts in $\mathcal{C}$ is itself in the collection $\mathcal{C}$.

Note: Condition iii) is sometimes omitted, in light of the following fact: Given set $M$ and collection $\mathcal{C}$ of $n$-charts satisfying i) and ii) above, then $(M, \hat{\mathcal{C}})$, where $\hat{\mathcal{C}}$ is the collection of all $n$-charts on $M$ compatible with all those in $\mathcal{C}$, satisfies all three conditions above, i.e., is a manifold as here defined. Let us agree that, hereafter, whenever we say “$n$-manifold”, then “$n$ is a non-negative integer” is implied.

Let $(M, \mathcal{C})$ be an $n$-manifold. A (smooth) function on $M$ is a map $M \xrightarrow{f} \mathbb{R}$ with the following property: For every chart $(U, \phi)$ in the collection $\mathcal{C}$, the map $\phi[U] \xrightarrow{f \circ \phi^{-1}} \mathbb{R}$ is smooth.

Note: Each constant function on $M$ is smooth, as is the pointwise sum and product of any two smooth functions.
Let \((M, C)\) be an \(n\)-manifold. A **(smooth) curve** on \(M\) is a map \(R \xrightarrow{\gamma} M\) with the following property: For every chart \((U, \phi)\) in the collection \(C\), i) \(\gamma^{-1}[U]\) is open in \(R\), and ii) the mapping \(\gamma^{-1}[U] \xrightarrow{\phi \circ \gamma} \mathbb{R}^n\) is smooth.

Let \((M, C)\) be an \(n\)-manifold, and let \(p \in M\). A **tangent vector** at \(p\) is a mapping \(F : \mathbb{R} \to M\), where \(F\) denotes the collection of all smooth functions on this manifold, such that: i) for any constant function, \(\xi(f) = 0\); and for \(f\) and \(g\) any two smooth functions, ii) \(\xi(f + g) = \xi(f) + \xi(g)\), and iii) \(\xi(fg) = f(p)\xi(g) + g(p)\xi(f)\).

Note: The set of tangent vectors at \(p\) has the structure of an \(n\)-dimensional vector space (under the obvious definitions of addition and scalar multiplication of tangent vectors).

Let \((M, C)\) be an \(n\)-manifold, \(R \xrightarrow{\gamma} M\) a smooth curve on this manifold, and \(\lambda_0\) a real number. Set \(p = \gamma(\lambda_0) \in M\). The **tangent** to \(\gamma\) at \(\lambda_0\) is the tangent vector \(\xi\) at \(p\) such that: For every smooth function \(f\) on \(M\), \(\xi(f) = d(f \circ \gamma)/d\lambda\big|_{\lambda_0}\) (noting that the right side does indeed define a tangent vector).

Let \((M, C)\) be an \(n\)-manifold, \(p \in M\), \(\xi\) a tangent vector at \(p\), and \((U, \phi)\) an \(n\)-chart in the collection \(C\), such that \(p \in U\). The **components** of \(\xi\) with respect to this chart are the \(n\) numbers \((\xi^1, \cdots, \xi^n)\) such that: For any smooth function \(f\) on \(M\),

\[
\xi(f) = \sum_{i=1}^{n} \xi^i \left[ \frac{\partial(f \circ \phi^{-1})}{\partial x^i} \right] \big|_{\phi(p)}.
\]  

Let \((M, C)\) be an \(n\)-manifold, and let \(p \in M\). A **covector** at \(p\) is a linear mapping \(T \xrightarrow{\mu} R\), where \(T\) is the \(n\)-dimensional vector space of tangent vectors at \(p\).

Note: The set of covectors at \(p\) has the structure of an \(n\)-dimensional vector space (under the obvious definitions of addition and scalar multiplication of covectors).

Let \((M, C)\) be an \(n\)-manifold, let \(f\) be a smooth function on \(M\), and let \(p \in M\). The **gradient** of \(f\) at \(p\) is the covector, written \(\nabla f\), at \(p\) with the
following action: For any tangent vector $\xi$ at $p$, $(\nabla f)(\xi) = \xi(f)$ (noting that the right side does indeed define a covector).

Let $(M, C)$ be an $n$-manifold, $p \in M$, $\mu$ a covector at $p$, and $(U, \phi)$ an $n$-chart in the collection $C$, such that $p \in U$. The components of $\mu$ with respect to this chart are the $n$ numbers $(\mu_1, \cdots, \mu_n)$ such that: For any tangent vector $\xi$ at $p$,

$$\mu(\xi) = \sum_{i=1}^{n} \mu_i \xi^i,$$

(2)

where $(\xi^1, \cdots, \xi^n)$ are the components of $\xi$ with respect to this chart.

Let $(M, C)$ be an $n$-manifold. A tangent vector field (resp, covector field) on $M$ is a mapping that sends each point $p$ of $M$ to a tangent vector (resp, covector) at $p$. Fix such a field $\xi$ (resp, $\mu$), and let $(U, \phi)$ be an $n$-chart in the collection $C$. Consider the following map from the open set $\phi[U]$ in $\mathbb{R}^n$ to $\mathbb{R}^n$: It sends $(x^1, \cdots, x^n) \in \phi[U]$ to the components of $\xi$ (resp, $\mu$), evaluated at $\phi^{-1}(x^1, \cdots, x^n)$. We say that the tangent vector field $\xi$ (resp, covector field $\mu$) is smooth if this mapping is smooth.

Let $(M, C)$ be an $n$-manifold, and $f$ a smooth function on $M$. The gradient of $f$ is the smooth covector field (also written $\nabla f$) on $M$ whose value at each $p \in M$ is the gradient of $f$ at $p$.

Let $(M, C)$ be an $n$-manifold, and $\mu$ a smooth covector field on $M$. This field is said to be exact if there exists a smooth function $f$ on $M$ such that $\mu = \nabla f$.

2 Measure Theory

Let $X$ be a set, and $X \overset{f}{\rightarrow}$ real-valued function on $X$. Then we say that $\Sigma_X f$ converges to $a \in \mathbb{R}$ provided: Given any $\epsilon > 0$, there exists a finite $Y$, $Y \subseteq X$ such that, for any finite $Y'$, $Y \subseteq Y' \subseteq X$, $|\Sigma_{Y'} f - a| \leq \epsilon$. Here, the sum of $f$ over a finite set has the obvious meaning: the sum of the values of $f$ on that set.

Note: This is what (absolute) convergence of a sum really is. We may replace the reals, above, by the complexes (or, indeed, by any abelian topological group). Note that we can, potentially, “sum” over even an uncountable
set $X$! [If you know the definition of a net, you can formulate the above even more elegantly: Let the directed set be that of the finite subsets of $X$.]

A **measurable space** is a set $S$, together with a nonempty collection, $\mathcal{S}$ of subsets of $S$, satisfying the following two conditions:

1. For any $A, B$ in the collection $\mathcal{S}$, the set $A - B$ is also in $\mathcal{S}$.
2. For any $A_1, A_2, \ldots \in \mathcal{S}$, $\bigcup A_i \in \mathcal{S}$.

Note: Sometimes the additional condition $S \in \mathcal{S}$ is included in which case $(S, \mathcal{S})$ is called a $\sigma$--**algebra**.

Let $(S, \mathcal{S})$ be a measurable space. A **measure** on $(S, \mathcal{S})$ consists of a nonempty subset of $\mathcal{S}$, $M \subseteq \mathcal{S}$, together with a mapping $M \to \mathbb{R}^+$ (where $\mathbb{R}^+$ denotes the set of nonnegative reals), satisfying the following two conditions:

1. For any $A \in M$ and any $B \subseteq A$, with $B \in \mathcal{S}$, we have $B \in M$.
2. Let $A_1, A_2, \ldots \in M$ be disjoint, and set $A = A_1 \cup A_2 \cup \cdots$. Then: This union $A$ is in $M$ if and only if the sum $\mu(A_1) + \mu(A_2) + \cdots$ converges; and when these hold that sum is precisely $\mu(A)$.

Fix any subset of the reals, $X \subseteq \mathbb{R}$. Let $I_1, I_2, \cdots$ be any countable collection of open intervals (not necessarily disjoint) in $\mathbb{R}$ such that $X \subseteq I_1 \cup I_2 \cup \cdots$ (noting that there exists at least one such collection); and denote by $m$ the sum of the lengths of these intervals (possibly $\infty$). By the **outer measure** of $X$, $\mu^*(X)$, we mean the greatest lower bound, over all such collections, of $m$. Thus, $\mu^*(X)$ is a non-negative number, or possibly “$\infty$”.

For $X$ and $Y$ be any two subsets of the reals, $R$, set $d(X, Y) = \mu^*(X - Y) + \mu^*(Y - X)$, so $d(X, Y)$ is a nonnegative or the symbol “$\infty$”.

Denote by $\mathcal{M}$ the collection of all subsets $A$ of $S = \mathbb{R}$ with the following property: Given any $\epsilon > 0$, there exists a $K \subseteq \mathbb{R}$, where $K$ is a finite union of open intervals, such that $d(A, K) \leq \epsilon$. And, for $A \in \mathcal{M}$, set $\mu(A) = \mu^*(A)$, a nonnegative number. A set $A \subseteq \mathbb{R}$ is said to be **Lebesgue measurable** if it is a countable union of sets in $\mathcal{M}$; and, for $A \in \mathcal{M}$, the number $\mu(A)$ is called the **Lebesgue measure** of $A$.

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1By $A - B$, we mean $A \cap B^c$, i.e., the set of all points of $A$ that are not in $B$. 

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Let \((S, \mathcal{S})\) be a measurable space, and \(S \xrightarrow{f} R\) a function on \(S\). We say that this \(f\) is \textit{measurable} if, for every open \(O \subset R\), \(f^{-1}[O] \in \mathcal{S}\).

Let \((S, \mathcal{S})\) be a measurable space. By a \textit{step function} on \(S\) we mean a measurable, real-valued function on \(S\) with finite range.

Let \((S, \mathcal{S}, \mathcal{M}, \mu)\) be a measure space, with \(S \in \mathcal{M}\), and let \(f\) be a step function (with values \(a_1, \ldots, a_s\) on disjoint sets \(A_1, \ldots, A_s \in \mathcal{M}\), where \(A_1 \cup \cdots \cup A_s = S\)) on \(S\). The \textit{integral} of \(f\) (over \(S\), with respect to \(\mu\)) is the number \(\int_S f \, d\mu = a_1 \mu(A_1) + \cdots + a_s \mu(A_s)\).

Let \((S, \mathcal{S}, \mathcal{M}, \mu)\) be a measure space, with \(S \in \mathcal{M}\), and let \(f\) be a bounded measurable function on \(S\). Then the \textit{integral} of \(f\) (over \(S\), with respect to \(\mu\)) is the number \(\int_S f \, d\mu = \lim_{\epsilon \to 0} \int_S g_\epsilon \, d\mu\), where the \(g_\epsilon\) are step functions such that \(|f - g_\epsilon| \leq \epsilon\).

Let \((S, \mathcal{S}, \mathcal{M}, \mu)\) be a measure space (not necessarily finite), and let \(f \geq 0\) be a measurable function (not necessarily bounded) on \(S\). We say that \(f\) is \textit{integrable} over \(A \in \mathcal{S}\) provided there exist disjoint sets \(A_1, A_2, \cdots\) each of finite measure and on each of which \(f\) is bounded, such that \(\cup A_i = A\) and \(\sum(f_{A_i} \, d\mu)\) converges. We write \(\int_A f \, d\mu\) for that sum.

Let \((S, \mathcal{S}, \mathcal{M}, \mu)\) be a measure space, and \(f\) a measurable function (not necessarily nonnegative) on \(S\). We say that \(f\) is \textit{integrable} over \(A \in \mathcal{S}\) provided each of the (nonnegative, measurable) functions \(f^+ = (1/2)(|f| + f)\) and \(f^- = (1/2)(|f| - f)\) is integrable over \(A\). We write \(\int_A f \, d\mu\) for the difference of these two integrals.

3 Hilbert Spaces

Recall that a \textit{complex vector space} is a set \(H\), together with a mapping \(H \times H \xrightarrow{\text{add}} H\) (called \textit{addition}, and written \(x+y\)) and a mapping \(C \times H \xrightarrow{\text{mult}} H\) (called \textit{scalar multiplication}, and written \(cx\)), satisfying the nine or so standard vector-space conditions. An \textit{inner product} on a complex vector space \(H\) is a mapping \(H \times H \xrightarrow{\text{prod}} C\) (called the \textit{inner product}, and written
\langle x | y \rangle \) which i) is antilinear in the first entry and linear in the second (i.e., satisfies \( \langle x + cz | y \rangle = \langle x | y \rangle + \overline{c} \langle z | y \rangle \) and \( \langle x | y + cz \rangle = \langle x | y \rangle + c \langle x | z \rangle \), for any \( x, y, z \in H \) and \( c \in C \); ii) complex-conjugates under order-reversal (i.e., satisfies \( \langle x | y \rangle = \overline{\langle y | x \rangle} \) for any \( x, y \in H \); and iii) is such that \( \langle x | x \rangle \geq 0 \), with equality if and only if \( x = 0 \). Fix any complex vector space \( H \) with inner product \( \langle \cdot | \cdot \rangle \). Then, for \( x \in H \), we set \( \| x \| = \langle x | x \rangle^{1/2} \) (called the norm of \( x \)). A sequence \( x_1, x_2, \cdots \in H \) is called a Cauchy sequence if, for every \( \epsilon > 0 \), there exists a number \( N \) such that \( \| x_n - x_m \| \leq \epsilon \) whenever \( n, m \geq N \). A sequence \( x_1, x_2, \cdots \in H \) is said to converge to \( x \in H \) provided \( \| x - x_n \| \to 0 \) as \( n \to \infty \). A complex vector space with inner product is said to be complete if every Cauchy sequence therein converges to some element of \( H \).

A Hilbert space is a complex vector space with inner product that is complete. [We frequently write \(|x\rangle\) to indicate that \( x \) is an element of Hilbert space \( H \).]

Let \( H \) be a Hilbert space. Element \(|x\rangle\) is said to be unit provided \( \| x \| = 1 \); and elements \(|x\rangle, |y\rangle\) are said to be orthogonal provided \( \langle x | y \rangle = 0 \). We say that \( \sum |x_m \rangle \), where the sum is over \( m = 0, 1, \cdots \), converges to \(|x\rangle\) provided the partial sums, \( \sum_1^n |x_m \rangle \), converge (as defined above) to \( x \).

Let \( H \) be a Hilbert space. A collection \(|x_{\gamma}\rangle \) (where \( \gamma \) ranges through some indexing set \( \Gamma \)) is called a basis for \( H \) provided i) each \(|x_{\gamma}\rangle\) is unit; and any two \(|x_{\gamma}\rangle\), for distinct \( \gamma \), are orthogonal; and ii) for every \(|x\rangle\) there exist complex numbers \( c_{\gamma} \) such that \( \sum_{\Gamma} c_{\gamma} |x_{\gamma}\rangle \) converges to \( x \).

Let \((S, \mathcal{S}, \mathcal{M}, \mu)\) be a measure space. By \( L^2(S, \mathcal{S}, \mathcal{M}, \mu) \) we mean the Hilbert space whose elements are complex-valued, measurable, square-integrable functions \( f \) on \( S \), with two functions identified if they differ on a set of measure zero; whose addition and scalar multiplication are pointwise addition and scalar multiplication of the functions; and whose inner product is given by \( \langle f | g \rangle = \int_S f \overline{g} \, d\mu \) (noting that these operations are independent of representative; and that this last integral converges).
4 Operators

Let $H$ be a Hilbert space. A (bounded) operator on $H$ is a complex-linear map, $H \xrightarrow{A} H$, such that, for some number $a > 0$, $\| Ax \| \leq a \| x \|$ for every $x$. The greatest lower bound of such $a$’s, written $|A|$, is called the norm of operator $A$.

Let $H$ be a Hilbert space, and $A$ a (bounded) operator on $H$. Let $c$ be a complex number, and $|x\rangle$ a nonzero vector in $H$, such that $A|x\rangle = c|x\rangle$. Then we say that $|x\rangle$ is an eigenvector of $A$, with eigenvalue $c$. Fix $c$, and consider the collection consisting of all eigenvectors of $A$ with eigenvalue $c$ together with the zero vector. This collection is a subspace of $H$, called the eigenspace of $A$ (corresponding to eigenvalue $c$).

A (bounded) operator $A$ on $H$ is said to be self-adjoint provided: For every $|x\rangle, |y\rangle \in H$, $\langle x | Ay \rangle = \langle y | Ax \rangle$.

A (bounded) operator $P$ on $H$ is said to be a projection if $P$ is self-adjoint and satisfies $P \circ P = P$.

A (bounded) operator $U$ on $H$ is said to be unitary if $U$ is inner-product preserving (i.e., $\langle Ux | Uy \rangle = \langle x | y \rangle$, for every $|x\rangle, |y\rangle \in H$) and invertible (i.e., there exists a bounded operator $V$ on $H$ such that $U \circ V = V \circ U = I$).