Philosophy of Space, Time and Spacetime: Non-Euclidean Geometry and Curvature of Spaces

Dr. Erik Curiel

Munich Center For Mathematical Philosophy
Ludwig-Maximilians-Universität

Euclid's Five Postulates

1. To draw a straight line from any point to any point.
2. To produce a limited straight line in a straight line.
3. To describe a circle with any center and distance.
4. All right angles are equal to one another.
5. If one straight line falling on two straight lines makes the interior angles in the same direction less than two right angles, then the two straight lines, if produced *in infinitum*, meet one another in that direction in which the angles less than two right angles are.

Euclid's Fifth Postulate

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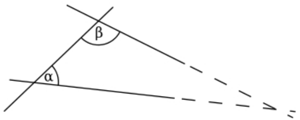


5-ONE *Simpler, equivalent formulation*: Given a line and a point not on the line, there is one line passing through the point parallel to the given line.

Significance of Postulate 5

- Contrast with Postulates 1-4
  - More complex, less obvious statement
  - Used to introduce parallel lines, *extendability* of constructions
  - Only axiom to refer to, rely on possibly infinite magnitudes
- Prove or dispense with Postulate 5?
  - Long history of attempts to prove Postulate 5 from other postulates, leads to independence proofs
  - Isolate the consequences of Postulate 5
  - Saccheri (1733), *Euclid Freed from Every Flaw*: attempts to derive absurd consequences from denial of 5-ONE

Alternatives for Postulate 5

5-ONE Given a line and a point not on the line, there is one line passing through the point parallel to the given line.

5-NONE Given a line and a point not on the line, there are no lines passing through the point parallel to the given line.

5-MANY Given a line and a point not on the line, there are many lines passing through the point parallel to the given line.

Geometrical Construction for 5-NONE

Reductio ad absurdum?

Saccheri's approach: assuming 5-NONE or 5-MANY (and other postulates) leads to contradictions, so 5-ONE must be correct.

- Construction: assuming 5-NONE, construct triangles with a common line as base
- Results: sum of angles of a triangle $> 180°$; circumference $\neq 2\pi R$

Non-Euclidean Geometries

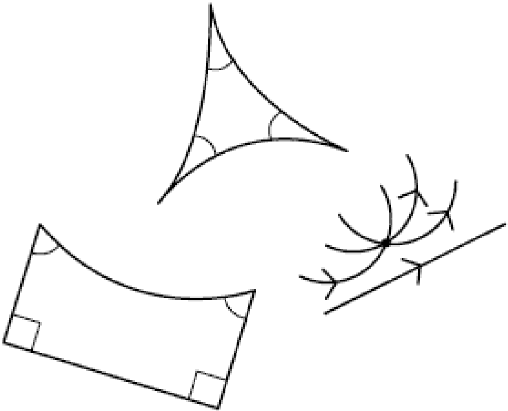
Pre-1830 (Saccheri et al.)

- Study alternatives to find *contradiction*
- Prove a number of results for "absurd" geometries with 5-NONE, 5-MANY

Nineteenth Century

- These are fully consistent *alternatives to Euclid*
- 5-NONE: spherical geometry
- 5-MANY: hyperbolic geometry

Hyperbolic Geometry



Consequences

5-??? What depends on choice of a version of postulate 5?

- Procedure:
  - Go back through *Elements*, trace dependence on 5-ONE
  - Replace with 5-NONE or 5 -MANY and derive new results
- Results: sum of angles of triangle $\neq 180^\circ$, $C \neq 2\pi r$, ...

Geometry of 5-NONE

What surface has the following properties?

- Pick an arbitrary point. Circles:
  - Nearby have $C \approx 2\pi R$
  - As R increases, $C < 2\pi R$
- Angles sum to more than Euclidean case (for triangles, quadrilaterals, etc.)
- True for every point $\rightarrow$ sphere

Geometry of 5-MANY

Properties of hyperboloid surface:

- "Extra space"
- Circumference $> 2\pi R$
- Angles sum to less than Euclidean case (for triangles, quadrilaterals, etc.)

Status of these Geometries?

How to respond to Saccheri et al., who thought a contradiction follows from 5-NONE or 5-MANY?

Relative Consistency Proof

If Euclidean geometry is consistent, *then* hyperbolic / spherical geometry is also consistent.
Proof based on "translation" Euclidean $\rightarrow$ non-Euclidean

Summary: Three Non-Euclidean Geometries

Geometry	Parallels	Straight Lines	Triangles	Circles
Euclidean	5-ONE	...	$180°$	$C = 2\pi R$
Spherical	5-NONE	finite	$> 180°$	$C < 2\pi R$
Hyperbolic	5-MANY	∞	$< 180°$	$C > 2\pi R$

Common Assumptions

Intrinsic geometry for surfaces of *constant curvature*.
Further generalization (Riemann): drop this assumption!

Geometry on a Surface

What does "geometry of figures drawn on surface of a sphere" mean?

- Intrinsic geometry
  - Geometry *on* the surface; measurements confined to the 2-dimensional surface
- Extrinsic geometry
  - Geometry of the surface as *embedded* in another space
  - 2-dimensional spherical surface *in* 3-dimensional space

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Importance of Being Intrinsic

- *Extrinsic* geometry useful . . .
- but limited in several ways:
  - Not all surfaces can be fully embedded in higher-dimensional space
  - Limits of visualization: 3-dimensional surface embedded in 4-dimensional space?
- So focus on *intrinsic* geometry instead

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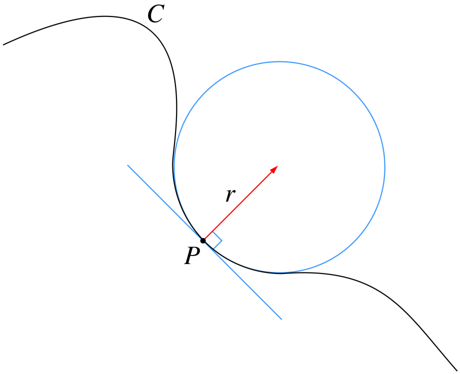
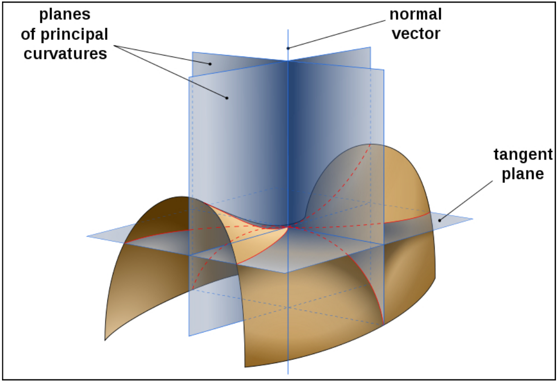
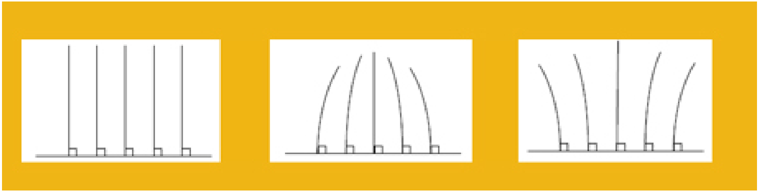
Curvature of a Line



Curvature of a Surface



Intrinsic Characterization of Curvature



Behavior of nearby initially parallel lines, reflects curvature

Non-Euclidean Geometries Revisited

Geometry	Parallels	Curvature	Geodesic Deviation
Euclidean	5-ONE	zero	constant
Spherical	5-NONE	positive	converge
Hyperbolic	5-MANY	negative	diverge

Riemannian Geometry

Curvature allowed to vary from point to point; link with geodesic deviation still holds.

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A Glossary for Riemann

magnitude-concept a measure of "size" used to quantify the magnitude of any instance of a given concept; "length", *e.g.*, is a magnitude-concept used to quantify spatial measurements; "number" is a magnitude-concept used to quantify the counting of discrete objects such as apples

mode of specification a unit or standard of magnitude, used to fix the "amount" of an instance of a given concept, as measured by the associated magnitude-concept; "meter", *e.g.*, is a mode of specification of spatial length, "5 meters" a fixed value of that mode; "dozen" is a mode of specification of the magnitude of a collection of apples; modes can vary either continuously (as for spatial lengths) or discretely (as for apples)

A Glossary for Riemann, Cont.

multiply extended magnitude a concept with an associated fixed number of magnitude-concepts, each of which must be specified according to its mode in order to individuate and identify an instance of that concept; ordinary physical space, *e.g.*, is a triply extended magnitude, because it needs three spatial lengths (coordinates, say, in a fixed coordinate system) to fix one of its points; the space of visual colors is also a triply extended magnitude, with three different types of magnitude-concepts, say, hue, saturation and brightness, that one must fix values for in order to fix an individual point in the space

A Glossary for Riemann, Cont.

manifold a collection of points or elements (objects, entities) that has the structure of a multiply extended magnitude, *i.e.*, has a fixed number of associated modes of specification of magnitude-concepts; the collection of all points of physical space is a 3-tuply extended, continuously varying manifold; the collection of all possible physical colors is as well, since an individual physical color can be uniquely identified by the values (modes of specification) of its hue, saturation and brightness, all of which vary continuously

A Glossary for Riemann, Cont.

measure relation on a manifold, a relation between pairs of points or elements that quantifies a notion of "distance", or "separation" more generally, between the pair (for *cognoscenti*: a Riemannian metric); correlatively or derivatively (depending on one's method of presentation), these relations also include other quantitative relations among geometrical objects living in the manifold, such as the angle between two intersecting curves (conformal structure), the volume of a solid figure (volume element), the intrinsic curvature of a curve, *etc.*

A Glossary for Riemann, Cont.

extension (or domain) relation a relation among points of a manifold that depends only on the modes of specification used to identify a point of the manifold, as opposed to a measure relation which imposes additional structure; unboundedness is an extension relation, because it is qualitative and not quantitative, as opposed to infinitude, which is a measure relation because it is quantitative; (for *cognoscenti*: the extension relations are the differential structure and topology of a manifold)