1. **Velocity Addition**

2. **Paradoxes**

3. **Spacetime and Spacetime Diagrams**
   - Basics
   - Tilt
   - Time Dilation
   - Length Contraction

4. **The Geometry of Minkowski Spacetime**
   - Space-Time Interval
   - Geodesics

5. **Paradoxes, Again**
   - Twin Paradox
Light Speed

- Special properties of the speed of light ... 
  1. All observers measure same speed for something moving at the speed of light (*not true* for other speeds)
  2. All observers agree whether something is moving *slower* or *faster* than $c$
  3. Speed limit: impossible to accelerate through speed of light

... must be reflected in new rule for adding velocities!
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Velocity Addition

Setup:
- Al and Bob as before; toy car moves on Bob’s train with velocity $u$ (according to Bob)
- What is velocity according to Al?
  - First guess: $u + v$
  - *But this is wrong!*
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Velocity Addition, 2

Velocity according to Al:

- Bob: “The little car is zooming around at speed $u$. How fast does it look to you?”

- Al: “Bob, your clocks are synchronized wrong and your distance measurements are off. So I can’t just add your $u$ to $v$…”

- Need to find “corrected” velocity, $u_{\text{corrected}}$
Velocity Addition, 2

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Velocity Addition, 3

Corrected Velocity

- Qualitatively: \( u_{corrected} = u \) (correction)
  
  Derive exact formula by considering how Al corrects Bob’s measurements \( \Rightarrow \) correction factor = \( \left( \frac{1 - \frac{v^2}{c^2}}{1 + \frac{uv}{c^2}} \right) \)

- Al’s final answer: \( u_{corrected} + V = \frac{u + v}{1 + \frac{uv}{c^2}} \leq u + v \)
Rocket Science

**Question:**

Rocket able to reach relative velocity $1/2c$
Why doesn’t a multiple stage rocket exceed $c$?

*Stage 1:* $v_r = 1/2c$

*Stage 2:* Reduction factor $= 5/4$, $v_r = (4/5)c$

*Stage 3:* Reduction factor $= 7/5$, $v_r = (13/14)c$

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Why doesn’t a multiple stage rocket exceed $c$?

Answer:
Boost provided by each stage contributes smaller change to relative velocity (as measured by earthbound observer)
“Einstein tax”: the reduction factor increases as the relative velocity increases, insuring that the rocket never reaches $c$
Summary: Velocity Addition

Why does the usual velocity addition rule fail?
- Velocity $u$ measured by Bob based on his rods and clocks
- Need to use $u_{\text{corrected}}$, leads to “reduction factor”

Why does it usually work just fine?
- For small relative velocities, the correction is extremely small and $u \approx u_{\text{corrected}}$

Special properties of light speed
- If $u < c$, then $u + v < c$
- If $u = c$, then $u + v = c$
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Refuting Relativity?

1. Relativity violates conceptual truths (e.g., time is absolute)
   - But what is the status of these truths?

2. Relativity conflicts with other theories of physics (e.g., gravity)
   - True! But can we formulate a new theory compatible with relativity . . . ?

3. Relativity predicts contradictory results
   - Version 1: Length contraction and time dilation effects are in fact contradictory
   - Version 2: Other more subtle contradictions (e.g., twin paradox)
Refuting Relativity?

- Bob measures the distance between the clocks and finds that it is 10 meters.
- Al measures a different value for the same distance.
- Conclusion: the theory leads to contradictory results!
General Response

- What is actually being measured?
  - Carefully specify the situation to determine what is being measured
  - Several *different* ways to measure lengths or time intervals between events
  - Relativity of simultaneity: source of disagreements regarding what is actually measured

- Different results for different quantities!
  - Contradiction *only apparent*; comparing *two different things*
Length Contraction

Al and Bob differ on distance between $P$ and $Q$:

- How do they measure the distance?
- Relativity of simultaneity: there is no procedure that they will both regard as correct way of measuring distance
Time Dilation

How to measure time dilation?

- Local comparison not sufficient (due to motion)
- Set up array of synchronized clocks along the train tracks
- Compare these to Bob’s watch
Time Dilation

Derive a contradiction?

- Al says moving clock runs slow...
- Bob says it runs at normal rate.
- But Bob thinks Al’s clocks aren’t synchronized correctly
At first, it seems that there is a contradiction. . .
Fails; relies on intuitions based on absolute simultaneity
General strategy to rebut “contradictions” and “paradoxes”:
- Implications of relativity of simultaneity
- Apparently contradictory accounts will agree on combination of spatial distances and time intervals; differences merely perspectival
- All accounts will agree on underlying, objective, invariant quantities, such as spatiotemporal interval, energy-momentum, electromagnetic field, etc.
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Minkowski’s Pronouncement (1908)

The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.
Space Diagrams

January

April

July

October
Spacetime Diagrams

- October
- July
- April
- January
Light Cone
### Terminology

- **timelike worldline**
- **lightlike worldline**
- **spacelike curve**
- **future light cone**
- **past light cone**
- **hypersurface of simultaneity**
- **event**

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**Spacetime and Spacetime Diagrams**

The Geometry of Minkowski Spacetime

Paradoxes, Again
Al and Bob in Spacetime
Al and Bob in Spacetime
Simultaneity Slices
Tilt

Moving Observers

Simultaneity Slices
- Time axis tilts *toward* 45°
- Spatial axes tilt by same amount
Explaining the Tilt
Objection: time and space not orthogonal!

Reply: contrast between \textit{normal} geometry and \textit{spacetime} geometry
Illustration: Time Dilation

- Observer Moving with Rod
  - Light beam bouncing back and forth
  - Equal time intervals
- Earthbound Observer
  - Not equal time intervals
Illustration: Time Dilation

Observer Moving with Rod

Earthbound Observer
Illustration: Time Dilation

Rod Observer according to Earthbound Observer
"Half-Twin" Effect

A's hypersurface of simultaneity

B's hypersurface of simultaneity

Time Dilation
Length Contraction
1 Velocity Addition

2 Paradoxes

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Newtonian Spacetime

All observers:
- agree on time
- ...agree on acceleration
- ...agree on simultaneous spatial distance
- ...disagree on non-simultaneous spatial distance
- ...disagree on velocities
- *No shift* in simultaneity slices

From Roger Penrose, *The Road to Reality*
Minkowski Spacetime

All observers: agree on spacetime interval

- . . . agree on acceleration
- . . . disagree on velocities
  except velocity of light
- . . . disagree on space, time distances
- *Shift* in simultaneity slices
Setup: Al and Bob, measuring distance from $P$ to $Q$
Distance in Euclidean Geometry

- Distance from $P$ to $Q$:
  - Ali: $PQ^2 = RQ^2 + PR^2$
  - Bob: $PQ^2 = SQ^2 + PS^2$

Agree on this distance
Space-Time Interval

Spacetime Distance (Interval)

- Spacetime interval from $P$ to $Q$:
  - Al: $PQ^2 = RQ^2 - PR^2$
  - Bob: $PQ^2 = SQ^2 - PS^2$
- Agree on this quantity
Space-Time Interval

**Spacetime vs. Euclidean Distance**

**Euclidean Distance**

\[(\text{distance})^2 = (x\text{-distance})^2 + (y\text{-distance})^2\]

- **Always positive,** zero only for overlapping points

**Spacetime Interval**

\[(\text{st-distance})^2 = c^2(\text{time-distance})^2 - (\text{space-distance})^2\]

- **Positive:** temporal distance > spatial distance, *inside* light cone ("time-like")
- **Zero:** on the light cone ("light-like" or "null")
- **Negative:** spatial distance > temporal distance, *outside* light cone ("space-like")
Geodesics in Euclidean Geometry

A geodesic is the curve connecting two points that has the shortest possible length.
Geodesics in Euclidean and Minkowskian Geometry

- Euclidean geometry
  - Geodesics are *straight lines*

- Minkowski geometry
  - What are the geodesics for Minkowski spacetime?
  - Focus on *timelike curves*: length of a timelike curve measured by time elapsed by the watch of observer moving along the trajectory
  - What is the *longest* timelike curve passing through two points?
**Geodesics**

**Saving Time Costs Money!**

- *Longest* curve between \( p \) and \( q \): straight line, inertial observer
- *Shorter* curves: accelerate along a zig-zag trajectory
  
  ...so saving time (*shorter curve, less elapsed time*) costs money (*rocket fuel*)
1 Velocity Addition

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5 Paradoxes, Again
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Setup
- Twin $A$ stays at home
- Twin $B$ goes on a rocket ship
Twin Paradox

Paradox?

- Twin A: "B’s clock runs slow. Therefore I should be younger at our reunion."
- Twin B: "A’s clock runs slow. Therefore I should be younger at our reunion."
Twin Paradox

Paradox Lost
Twin Paradox and Geodesics

- **Stay-at-home twin**
  - Inertial observer has trajectory of *maximum* length, *longest* time elapsed

- **Traveling twin**
  - Uses money (acceleration of the rocket) to buy time (*shorter* spacetime trajectory)