Einstein for Everyone
Lecture 4: Paradoxes; Minkowski Spacetime

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Light Speed

- Special properties of the speed of light . . .
  1. All observers measure same speed for something moving at the speed of light (*not true* for other speeds)
  2. All observers agree whether something is moving *slower* or *faster* than c
  3. Speed limit: impossible to accelerate through speed of light
- . . . must be reflected in new rule for adding velocities!

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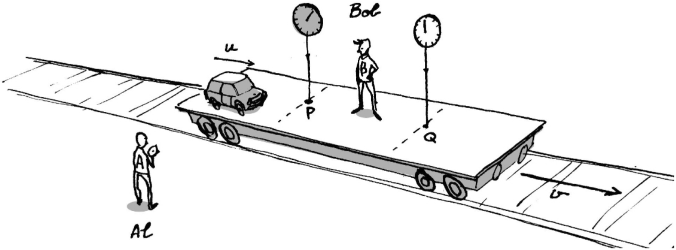
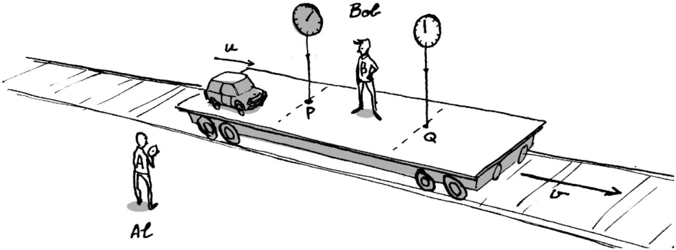
Velocity Addition



Setup:

- Al and Bob as before; toy car moves on Bob's train with velocity u (according to Bob)
- What is velocity according to Al?
  - First guess: $u + v$
- *But this is wrong!*

Velocity Addition



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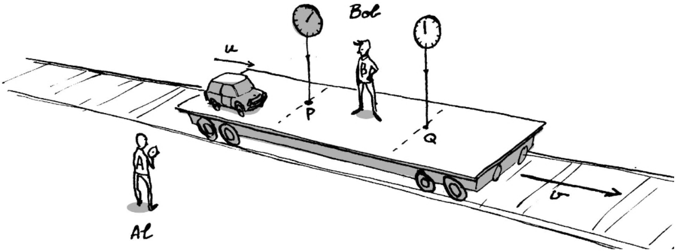
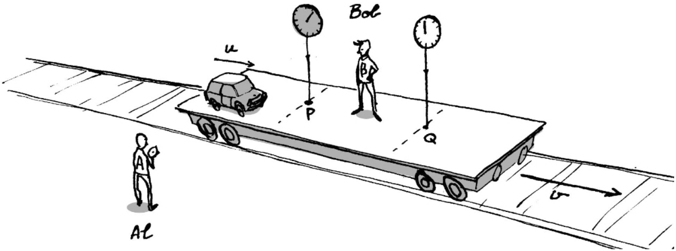


Velocity according to Al:

- Bob: "The little car is zooming around at speed u. How fast does it look to you?"
- Al: "Bob, your clocks are synchronized wrong and your distance measurements are off. So I can't just add your u to v..."
- Need to find "corrected" velocity, $u_{\text{corrected}}$

Velocity Addition, 2



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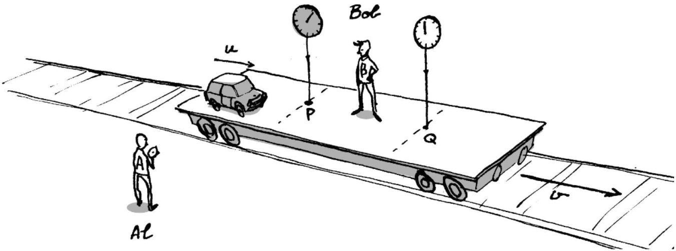
Velocity Addition, 3



Corrected Velocity

- Qualitatively: $u_{\text{corrected}} = u\,(\text{correction})$
  Derive exact formula by considering how Al corrects Bob's measurements $\Rightarrow$ correction factor $= \left(\frac{1-\frac{v^2}{c^2}}{1+\frac{uv}{c^2}}\right)$
- Al's final answer: $u_{\text{corrected}} + \mathrm{v} = \dfrac{u+v}{1+\frac{uv}{c^2}} \leq u+v$

Rocket Science

Question:

Rocket able to reach relative velocity $1/2c$
Why doesn't a multiple stage rocket exceed c?

Stage 1: $v_r = 1/2c$

Stage 2: Reduction factor $= 5/4$, $v_r = (4/5)c$

Stage 3: Reduction factor $= 7/5$, $v_r = (13/14)c$

Stage 4: Reduction factor $= 41/28$, $v_r = (40/41)c$

Stage 5: Reduction factor $= 61/41$, $v_r = (121/122)c$

... and so on ...

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Rocket Science

Question:

Rocket able to reach relative velocity $1/2c$
Why doesn't a multiple stage rocket exceed c?

Answer:

Boost provided by each stage contributes smaller change to relative velocity (as measured by earthbound observer)
"Einstein tax": the reduction factor increases as the relative velocity increases, insuring that the rocket never reaches c

Summary: Velocity Addition

- Why does the usual velocity addition rule fail?
  - Velocity u measured by Bob based on *his rods and clocks*
  - Need to use $u_{\text{corrected}}$, leads to "reduction factor"
- Why does it usually work just fine?
  - For small relative velocities, the correction is extremely small and $u \approx u_{\text{corrected}}$
- Special properties of light speed
  - If $u < c$, then $u + v < c$
  - If $u = c$, then $u + v = c$

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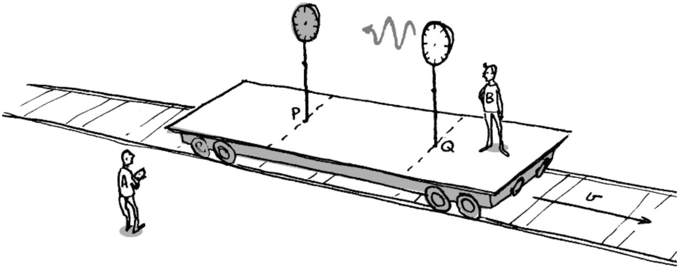
Refuting Relativity?

1. Relativity violates conceptual truths (e.g., time is absolute)
    - But what is the status of these truths?
2. Relativity conflicts with other theories of physics (e.g., gravity)
    - True! But can we formulate a new theory compatible with relativity . . . ?
3. Relativity predicts contradictory results
    - *Version 1*: Length contraction and time dilation effects are in fact contradictory
    - *Version 2*: Other more subtle contradictions (e.g., twin paradox)

Refuting Relativity?

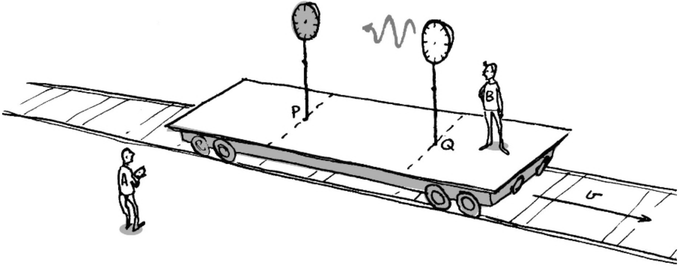


- Bob measures the distance between the clocks and finds that it is 10 meters
- Al measures a *different* value for the same distance
- *Conclusion*: the theory leads to contradictory results!

General Response

- What is actually being measured?
  - Carefully specify the situation to determine what is being measured
  - Several *different* ways to measure lengths or time intervals between events
  - Relativity of simultaneity: source of disagreements regarding what is actually measured
- Different results for different quantities!
  - Contradiction *only apparent*; comparing *two different things*

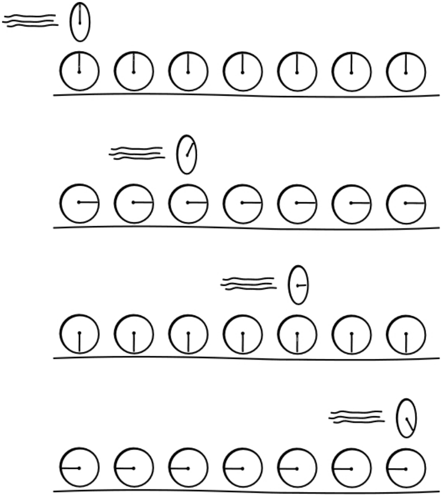
Length Contraction



Al and Bob differ on distance between P and Q:

- How do they measure the distance?
- Relativity of simultaneity: there is no procedure that they will both regard as correct way of measuring distance

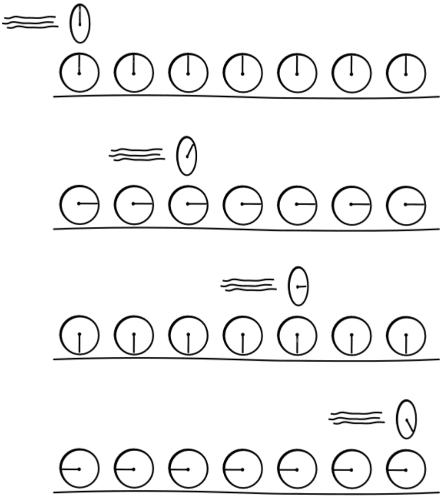
Time Dilation



How to measure time dilation?

- Local comparison not sufficient (due to motion)
- Set up array of synchronized clocks along the train tracks
- Compare these to Bob's watch

Time Dilation



Derive a contradiction?

- Al says moving clock runs slow. . .
- Bob says it runs at normal rate.
- But Bob thinks Al's clocks aren't synchronized correctly

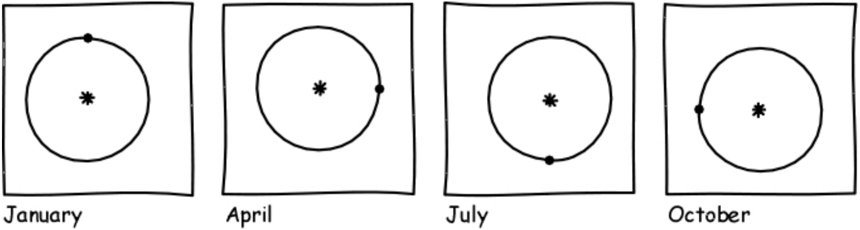
Summarizing

- Attempt to show that relativity is contradictory. . .
- Fails; relies on intuitions based on absolute simultaneity
- General strategy to rebut "contradictions" and "paradoxes":
  - Implications of relativity of simultaneity
  - Apparently contradictory accounts will *agree* on combination of spatial distances and time intervals; differences *merely perspectival*
  - All accounts will agree on underlying, objective, invariant quantities, such as spatiotemporal interval, energy-momentum, electromagnetic field, *etc*.

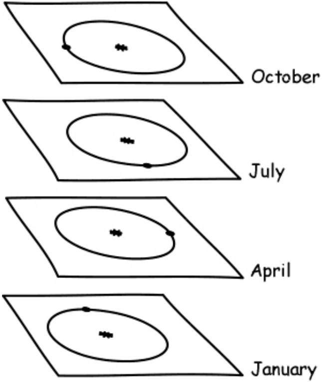
Minkowski's Pronouncement (1908)

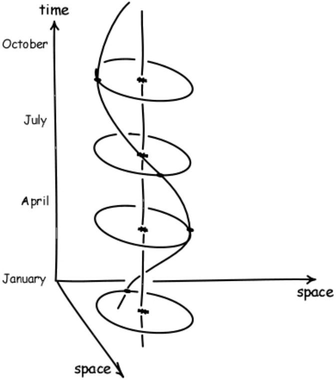
The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.

Space Diagrams

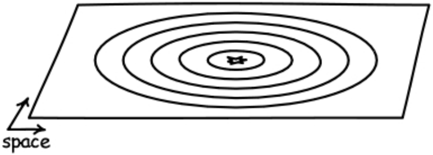


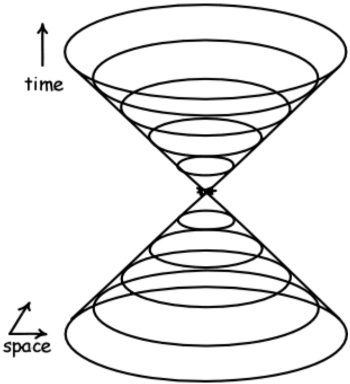
Spacetime Diagrams



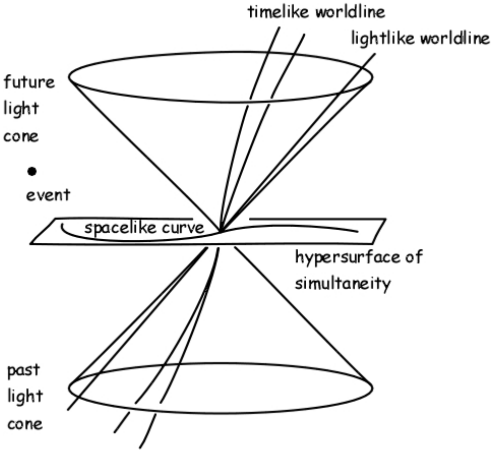


Light Cone

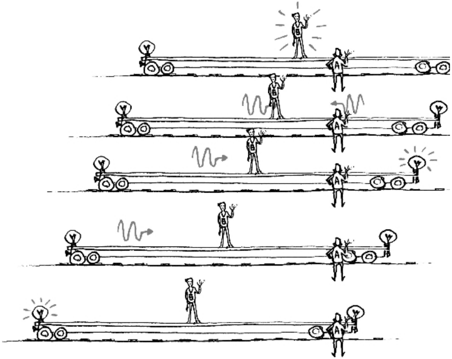


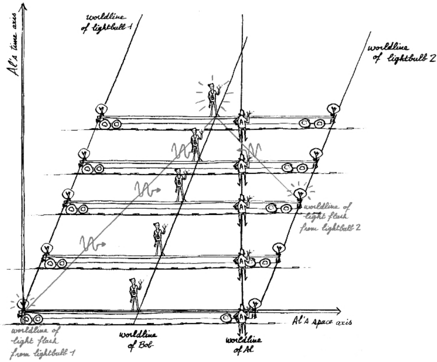


Terminology



Al and Bob in Spacetime





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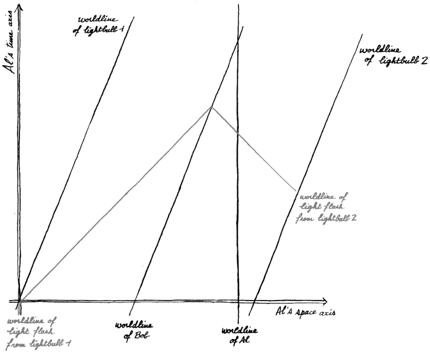
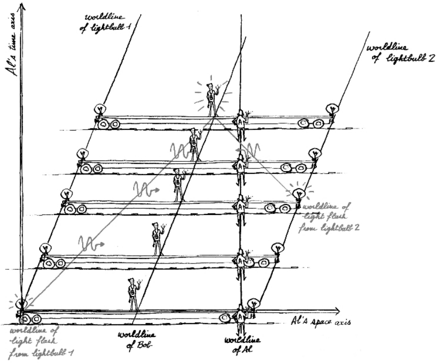
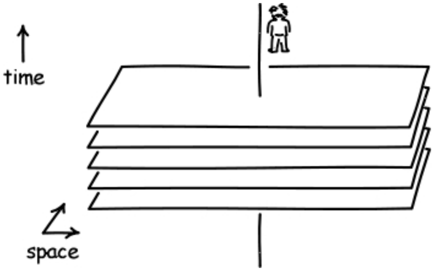
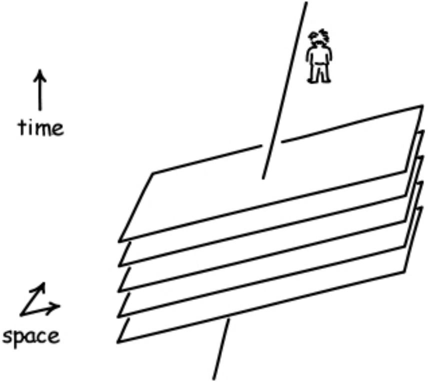


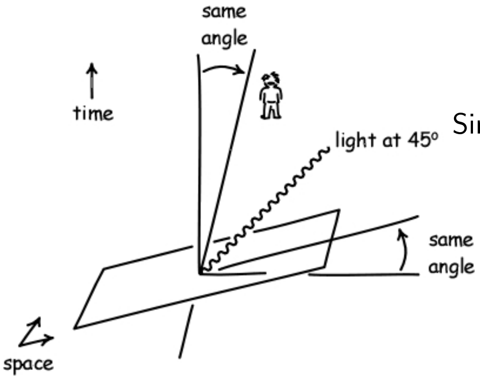


Simultaneity Slices





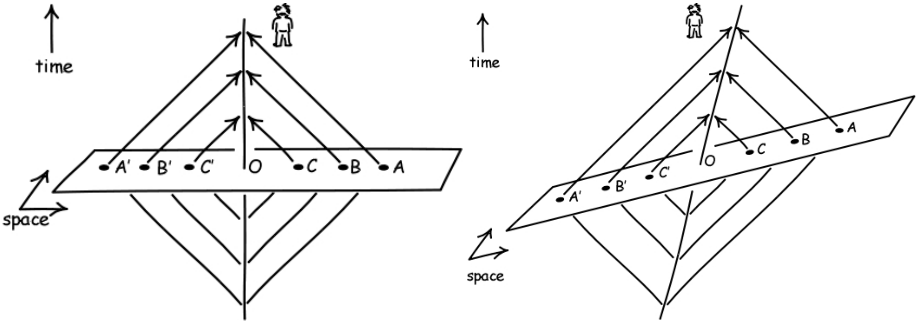
Moving Observers



Simultaneity Slices

- Time axis tilts *toward* 45°
- Spatial axes tilt by same amount

Explaining the Tilt



Explaining the Tilt, part 2

- Objection: time and space not orthogonal!
- Reply: contrast between *normal* geometry and *spacetime* geometry

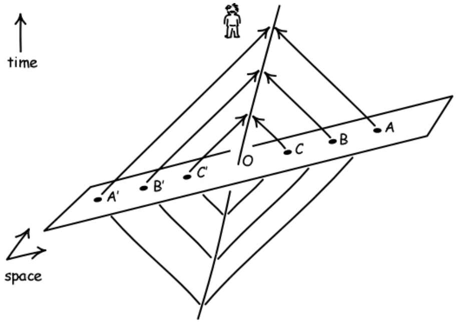


Illustration: Time Dilation

- Observer Moving with Rod
  - Light beam bouncing back and forth
  - Equal time intervals
- Earthbound Observer
  - Not equal time intervals

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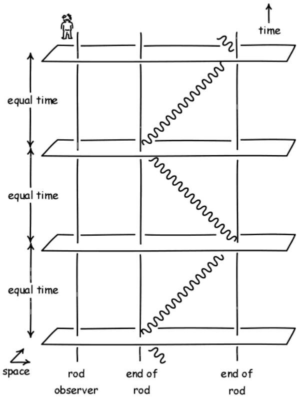
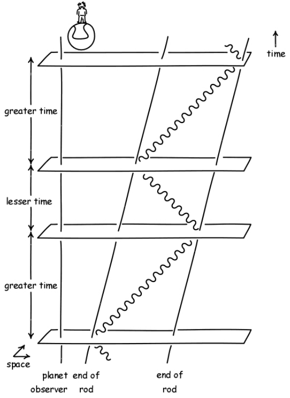
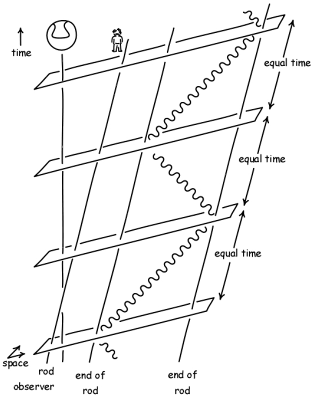


Observer Moving with Rod

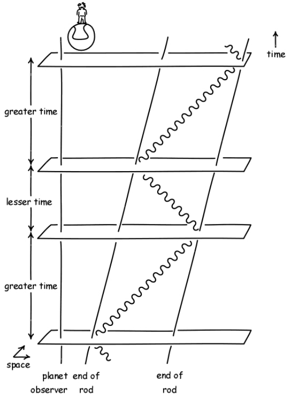


Earthbound Observer

Illustration: Time Dilation

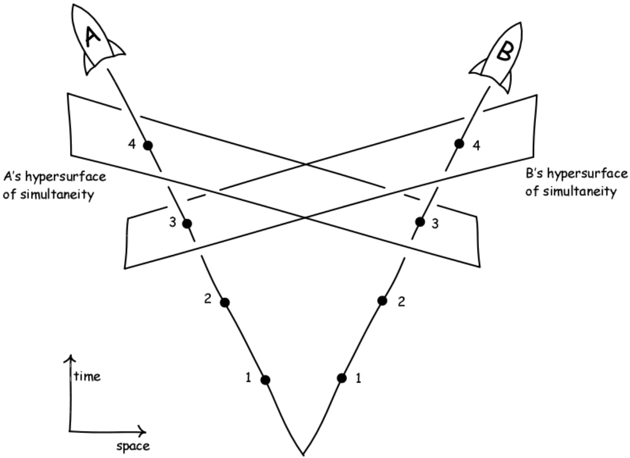


Rod Observer *according to Earthbound Observer*

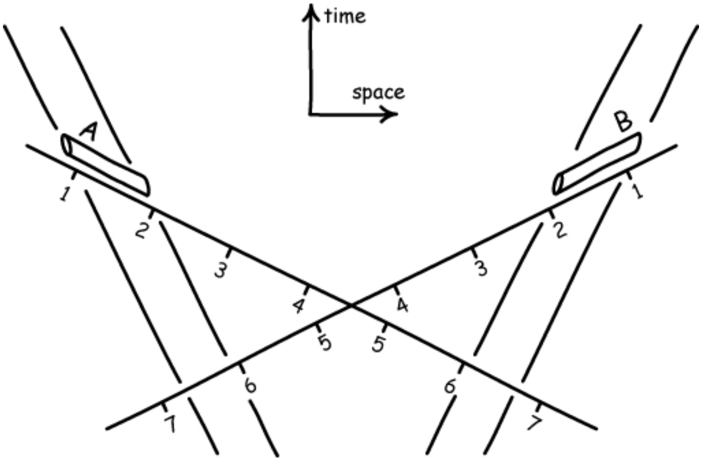


Earthbound Observer

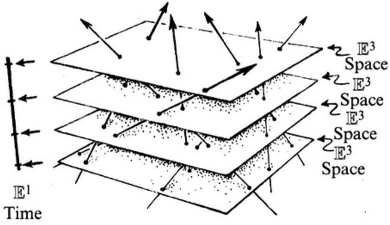
"Half-Twin" Effect



Length Contraction



Newtonian Spacetime

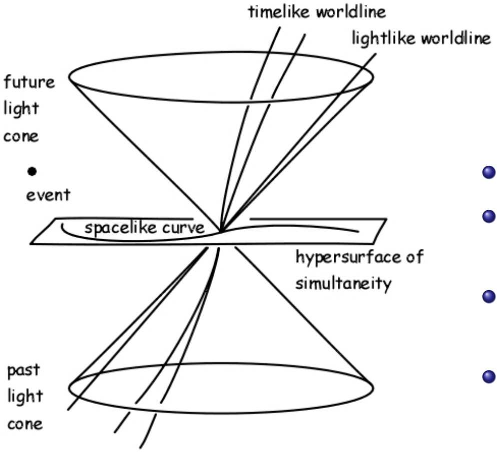


From Roger Penrose, *The Road to Reality*

All observers: agree on time

- . . . agree on acceleration
- . . . agree on simultaneous spatial distance
- . . . disagree on non-simultaneous spatial distance
- . . . disagree on velocities
- *No shift* in simultaneity slices

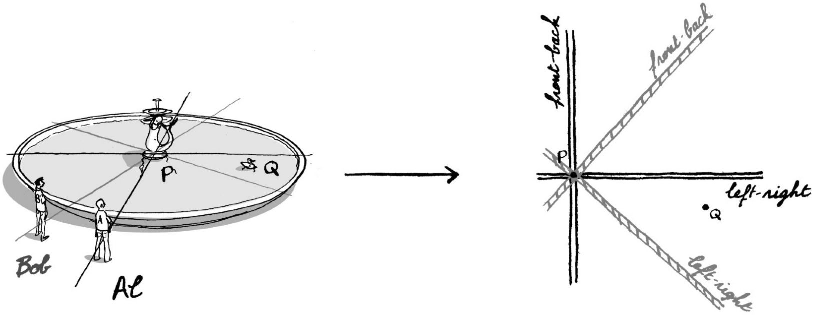
Minkowski Spacetime



All observers: agree on *spacetime interval*

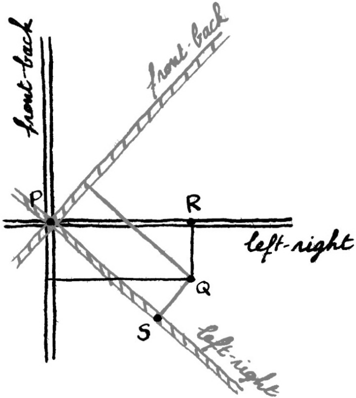
- . . . agree on acceleration
- . . . disagree on velocities *except velocity of light*
- . . . disagree on space, time distances
- *Shift* in simultaneity slices

Distance in Euclidean Geometry



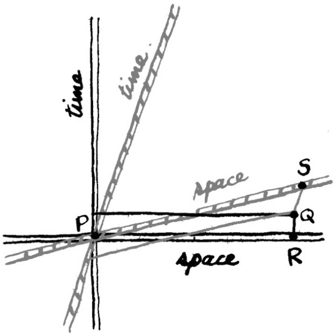
- Setup: Al and Bob, measuring distance from P to Q

Distance in Euclidean Geometry



- Distance from P to Q:
  Al: $PQ^2 = RQ^2 + PR^2$
  Bob: $PQ^2 = SQ^2 + PS^2$
- *Agree* on this distance

Spacetime Distance (Interval)



- Spacetime interval from P to Q:

  Al: $PQ^2 = RQ^2 - PR^2$

  Bob: $PQ^2 = SQ^2 - PS^2$
- *Agree* on this quantity

Spacetime vs. Euclidean Distance

Euclidean Distance

$$(\text{distance})^2 = (\text{x-distance})^2 + (\text{y-distance})^2$$

- *Always positive*, zero only for overlapping points

Spacetime Interval

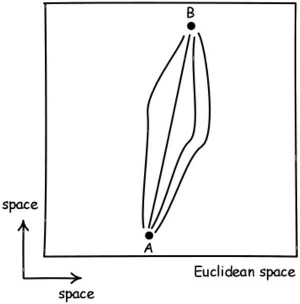
$$(\text{st-distance})^2 = c^2(\text{time-distance})^2 - (\text{space-distance})^2$$

- Positive: temporal distance > spatial distance, *inside* light cone ("time-like")
- Zero: *on* the light cone ("light-like" or "null")
- Negative: spatial distance > temporal distance, *outside* light cone ("space-like")

Geodesics in Euclidean Geometry

Geodesic

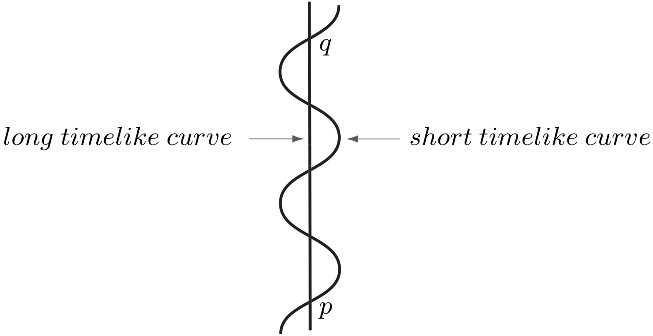
A *geodesic* is the curve connecting two points that has the *shortest* possible length.



Geodesics in Euclidean and Minkowskian Geometry

- Euclidean geometry
  - Geodesics are *straight lines*
- Minkowski geometry
  - What are the geodesics for Minkowski spacetime?
  - Focus on *timelike curves*: length of a timelike curve measured by time elapsed by the watch of observer moving along the trajectory
  - What is the *longest* timelike curve passing through two points?

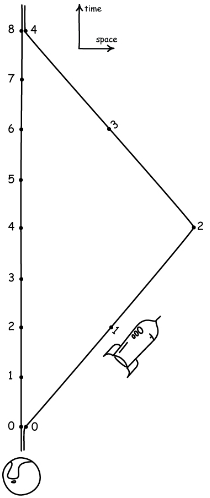
Saving Time Costs Money!



- *Longest* curve between p and q: straight line, inertial observer
- *Shorter* curves: accelerate along a zig-zag trajectory

  . . . so saving time (*shorter curve, less elapsed time*) costs money (*rocket fuel*)

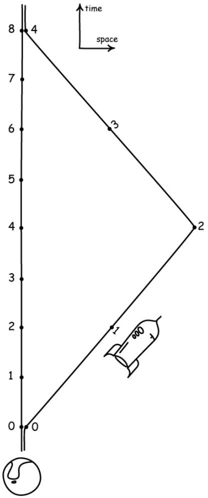
Twin Paradox



Setup

- Twin A stays at home
- Twin B goes on a rocket ship

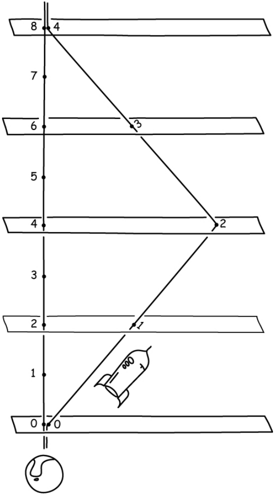
Twin Paradox

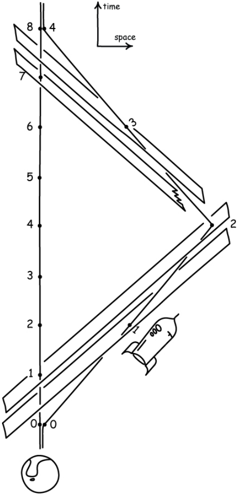


Paradox?

- Twin A: "B's clock runs slow. Therefore I should be younger at our reunion."
- Twin B: "A's clock runs slow. Therefore I should be younger at our reunion."

Paradox Lost





Twin Paradox and Geodesics

- Stay-at-home twin
  - Inertial observer has trajectory of *maximum* length, *longest* time elapsed
- Traveling twin
  - Uses money (acceleration of the rocket) to buy time (*shorter* spacetime trajectory)